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Cluster growth in the dynamical Erdős-Rényi process with forest fires

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Abstract

We investigate the growth of clusters within the forest fire model of Ráth and Tóth [24]. The model is a continuous-time Markov process, similar to the dynamical Erdős-Rényi random graph but with the addition of so-called *fires*. A vertex may catch fire at any moment and, when it does so, causes all edges within its connected cluster to burn, meaning that they instantaneously disappear. Each burned edge may later reappear.

We give a precise description of the process C_t of the size of the cluster of a tagged vertex, in the limit as the number of vertices in the model tends to infinity. We show that C_t is an explosive branching process with a time-inhomogeneous offspring distribution and instantaneous return to 1 on each explosion. Additionally, we show that the characteristic curves used to analyse the Smoluchowski-type coagulation equations associated to the model have a probabilistic interpretation in terms of the process C_t.

Keywords: Erdős-Rényi random graph ; forest fire ; self-organized criticality ; Smoluchowski coagulation equation.
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1 Introduction

Forest fire models are stochastic interacting particle systems in which the vertices or edges of a graph are gradually switched on, forming growing connected clusters. This growth is counterbalanced by so-called fires; each fire involves the rapid destruction of a single cluster by the switching off of its edges or vertices. Each fire is caused by the random spontaneous ignition of a single vertex, which we will call a lightning strike. The lightning strikes occur independently of the state of the system and are typically taken to be rare events so that on average fires are large.

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The evolution of a forest fire model is thus controlled by two competing forces, one that causes clusters to grow slowly and another that causes clusters to burn suddenly. One consequence is that a regime may exist in which the system exhibits self-organized criticality. This means that it is driven by its own dynamics towards a stationary state in which these two opposing forces are precisely balanced. In this state clusters may grow very large before they burn, typified by a heavy-tailed distribution of cluster sizes. Note that the term 'self-organized criticality' is a heuristic description of a model's behaviour, rather than any specific criterion. See Preussner [22] for a wide ranging discussion of self-organized criticality.

The existence of self-organized criticality in a forest fire model with lightning strikes has been predicted on the lattice $\mathbb{Z}^d$ by Drossel and Schwabl [11]. Recently, Ráth and Tóth [24] introduced a closely related model, on the complete graph, for which they were able to prove that self-organized criticality occurs in the limit of large system size. It is this model that we study in the present paper; we refer to it as the *Erdős-Rényi forest fire model*. In both models it is generally accepted that some form of self-organized criticality occurs when the system size tends to infinity and the rate per site at which lightning strikes occur tends slowly to 0.

The results of Ráth and Tóth [24] are concerned with the limiting behaviour of $v_l(t)$, the fraction of vertices that belong to clusters of size $l \in \mathbb{N}$ at time t. Their analysis is based on the important observation that the $v_l(t)$ can be combined into an appropriate generating function $V(t,z)$ which then (in the limit) satisfies a Burgers control problem. In this article we paint a further level of detail into the limiting picture; we study the evolution of the size of the cluster of a tagged vertex chosen uniformly at random. We determine the limit of this process as the system size tends to infinity. We show that the limit is an explosive branching process with a time-inhomogeneous offspring distribution and instantaneous return to 1 on each explosion. Thus in the limit the cluster of our tagged vertex burns at the moment that it becomes infinite.

We describe the Erdős-Rényi forest fire model in detail, along with our own results, in Sections 1.1 and 1.2. We will discuss connections between our own model and other models in the mathematical forest fires literature in Section 1.4.

1.1 The Erdős-Rényi Forest Fire Model

We now describe the Erdős-Rényi forest fire model $(\mathcal{Z}_t^n)_{t\geq 0}$ introduced in [24]. Let $n \in \mathbb{N}$ and consider $[n] = \{1, 2, \ldots, n\}$ as a set of n labelled vertices. At time $t \in [0, \infty)$ the state of the model is described by a multigraph $\mathcal{Z}_t^n$ with vertex set $[n]$ and unoriented edges; we permit parallel edges and loops. The *cluster* of vertex $k \in [n]$ at time t, written $\mathcal{C}_t^n(k)$, is the connected component of $\mathcal{Z}_t^n$ containing vertex k, i.e. the set of $j \in [n]$ such that there is a path along edges of $\mathcal{Z}_t^n$ from k to j.

Given some (deterministic) initial condition the process $(\mathcal{Z}_t^n)$ evolves with the following dynamics:

- Each unordered pair (j,k) carries a *growth clock* which rings at rate $\frac{1}{n}$. When the growth clock for (j,k) rings we add an edge joining j to k (recall that we permit parallel edges and loops).
- Each vertex carries a *fire clock* which rings with rate λ_n where $\lambda_n \in (0, \infty)$. When this fire clock of vertex k rings, the cluster of k is *burned*: all edges between pairs of vertices in $\mathcal{C}_t^n(k)$ are instantaneously removed.

The growth and fire clocks of distinct edges and vertices are mutually independent. For technical reasons detailed in [24] the process $t \mapsto \mathcal{Z}_t^n$ is taken to be left-continuous with right limits. Consequently it is Markov with respect to the filtration $\mathcal{F}_t^n = \sigma(\mathcal{Z}_s : s \leq t)$.

For each $l = 1, \dots, n$ we define

$$v_l^n(t) = \frac{1}{n}\big|\{k \in [n] \,:\, |\mathcal{C}_t(k)| = l\}\big| \tag{1.1}$$

to be the fraction of vertices in $[n]$ that are in a cluster of size l at time t. We will think of each vertex as having mass $1/n$, so that the total mass in the system is 1 and v_l^n is the proportion of mass in clusters of size l.

The effect of the fires results in four different phases of behaviour, as identified in [24]. We restrict our attention to only one (the most interesting) of these phases, where the lightning occurs sufficiently often to prevent the formation of a giant component but also sufficiently rarely that clusters of any fixed finite size are not burned in the limit as $n \to \infty$. The phase in which this occurs is defined by the following assumption.

Assumption 1.1. *As $n \to \infty$, $\lambda_n \to 0$ and $n\lambda_n \to \infty$.*

Under Assumption 1.1, as $n \to \infty$ a cluster of any constant size k will see a fire at rate $k\lambda_n \to 0$; in other words not at all. However, a cluster which grows to be of size around $\frac{1}{\lambda_n}$ will see lightning at a non-negligible rate. In the process $\mathcal{Z}^n$, a cluster of size $k \in \mathbb{N}$ and a (distinct) cluster of size $j \in \mathbb{N}$ join together at rate $\frac{kj}{n}$ to form a cluster of size $k + j$. It follows that for each fixed k, as $n \to \infty$, we expect $v_k^n(t)$ to see an inflow of mass at rate approximately $\frac{k}{2}\sum_{l=1}^{k-1} v_l^n(t)v_{k-l}^n(t)$ and an outflow at rate approximately $kv_k^n(t)$. The approximations here neglect growth clocks of edges joining vertices within the same cluster of size k, and lightning strikes causing clusters of size k to burn, both of which are negligible in the limit as $n \to \infty$ and $\lambda_n \to 0$. In our main result, Theorem 1.7, we will exploit these observations, combined with Theorem 1.5 (which improves on the main result of [24] and gives a global description of the behaviour of $\mathcal{Z}^n$ as $n \to \infty$), to describe the evolution of the size of the cluster of a tagged vertex.

To understand how $v_l^n(t)$ behaves as $n \to \infty$ it is sensible first to examine the simpler case $\lambda_n = 0$ (i.e. no fires) with $v_l^n(0) = \mathbb{1}\{l = 1\}$, so that initially we start with only singletons. In this case, after ignoring multiple edges and loops, which does not affect the partition into clusters, $\mathcal{Z}_t^n$ reduces to the Erdős-Rényi random graph on $[n]$ in which each edge is present independently with probability $1 - e^{-t/n}$. It is well known that, in the limit as $n \to \infty$, $v_l^n(t) \to v_l(t)$, where $v_l(t)$ is given explicitly in (1.3) below, and the behaviour observed is the following:

- For $t \in [0, 1)$, $l \mapsto v_l(t)$ has an exponential tail and $\sum_{l=1}^{\infty} v_l(t) = 1$.
- At $t = 1$, $l \mapsto v_l(1)$ has a polynomial tail and $\sum_{l=1}^{\infty} v_l(t) = 1$.
- For $t > 1$, $l \mapsto v_l(t)$ has an exponential tail but $\sum_{l=1}^{\infty} v_l(t) < 1$. The reason for this is that a giant component, containing a positive proportion of the vertices, has formed and this component is not picked up by the $v_l^n(\cdot)$ as $n \to \infty$. As $t \to \infty$ this (unique) giant component gradually accumulates all the vertices, so that $\sum_{l=1}^{\infty} v_l(t) \to 0$.

In fact, as our description above of the cluster growth rates suggests, in this case the limit $t \mapsto (v_l(t))_{l=1}^{\infty}$ satisfies

$$\frac{dv_k(t)}{dt} = \frac{k}{2}\sum_{l=1}^{k-1} v_l(t)v_{k-l}(t) - kv_k(t) \tag{1.2}$$

for all $k \geq 1$. These equations together are known as the Flory coagulation equation with multiplicative kernel. The term $-kv_k(t)$ in (1.2) indicates that the finite clusters interact with the giant component after the gelation time. The closely related system of coupled ODEs in which the term $-kv_k(t)$ is replaced by $-kv_k(t)\sum_{l=1}^{\infty} v_l(t)$ is called the Smoluchowski equation. Informally this modification corresponds to not alllowing

clusters of any fixed finite size to interact with giant components. Note that in the context of the theory of polymerization both the Smoluchowski and Flory equations are often written in terms of the *concentrations* $v_k(t)/k$.

The unique solution to (1.2) with initial condition $v_k(0) = \mathbb{1}\{k = 1\}$, is given explicitly by the Borel distribution

$$v_k(t) = \frac{k^{k-1}}{k!} e^{-kt} t^{k-1}. \tag{1.3}$$

Let us return to Assumption 1.1, *which we assume from now on*. As we said above, this means that in the limit any giant component is killed instantaneously as soon as it appears. However, clusters of any constant size $k \in \mathbb{N}$ do not see fires as $n \to \infty$. As a result, (1.2) still holds for $k \geq 2$, but v_1 feels an influx of singletons caused by the fires. Such fires can only occur once enough time has passed for the environment to grow clusters of large size; this time is known as the *gelation time* T_{gel}. The time T_{gel} depends on the initial condition $v_l(0) = \lim_{n\to\infty} v_l^n(0)$ and (see Section 3.2) is given by

$$T_{gel} = \left(\sum_{l=1}^{\infty} l v_l(0) \right)^{-1}. \tag{1.4}$$

Consequently, it is natural to expect that (1.2) holds for all k up until T_{gel}, whereas after T_{gel} (1.2) holds *only* for $k \geq 2$.

In [24] considerable effort is devoted to showing that (under Assumption 1.1) the limiting process $t \mapsto (v_l(t))_{l=1}^{\infty}$ satisfies $\sum_{l=1}^{\infty} v_l(t) = 1$ for all $t \geq 0$, in contrast to the Erdős-Rényi case. The result is that this equation replaces the $k = 1$ case of (1.2), for all time. To be precise, the system of equations we are interested in as the limit of the $v_k^n(\cdot)$s is described by the following result.

Theorem 1.2 (Ráth and Tóth [24])**.** *Suppose that $\sum_1^{\infty} l^3 v_l(0) < \infty$. Then there is a unique solution to the following system of equations, called the critical forest fire equations:*

$$\frac{dv_k(t)}{dt} = -k v_k(t) + \frac{k}{2} \sum_{l=1}^{k-1} v_l(t) v_{k-l}(t) \qquad \text{for } k \geq 2 \tag{1.5}$$

$$\sum_{l=1}^{\infty} v_l(t) = 1. \tag{1.6}$$

For such a solution, $v_k(t) \in [0, 1]$ for all $t \geq 0$ and all $k \in \mathbb{N}$. Further, there exists a function $\varphi : [0, \infty) \to \mathbb{R}$ such that for all $t \neq T_{gel}$ we have

$$\frac{dv_1(t)}{dt} = -v_1(t) + \varphi(t). \tag{1.7}$$

The function v_1 is continuous on $[0, \infty)$ and continuously differentiable on $(0, T_{gel}) \cup (T_{gel}, \infty)$. In fact, $\varphi = 0$ on $[0, T_{gel})$ and φ is both positive and locally Lipschitz on $[T_{gel}, \infty)$.

From the same initial conditions, the solution of (1.5)+(1.6) *coincides with the solution of* (1.2) *for $t \in [0, T_{gel}]$. For times $t > T_{gel}$ the solutions do not coincide. For $t \geq T_{gel}$ the solution of* (1.5)+(1.6) *satisfies $\sum_{l=k}^{\infty} v_l(t) \sim \sqrt{\frac{2\varphi(t)}{\pi}} k^{-1/2}$ as $k \to \infty$.*

Remark 1.3. Note that the functions (v_l) do not depend on (λ_n), except through Assumption 1.1. For each fixed n the random functions v_l^n do depend on λ_n, and finer analysis would be needed to see this dependence in the limit.

Remark 1.4. The function φ is the limiting rate at which mass within $\mathcal{Z}^n$ burns as $n \to \infty$, where each vertex is thought of as having mass $1/n$. Note that φ is not continuous at T_{gel}.

Note that the Flory equations (1.2) can be solved separately for $k = 1, 2, \ldots$ in turn; consequently existence and uniqueness of solutions is not difficult to prove. However, the critical forest fire equations (1.5)+(1.6) form a genuinely infinite system that is significantly harder to work with. As was observed in [24], equations (1.5)+(1.6) can be recast (using a suitable moment generating function) as a Burgers control problem (see equation (3.1)), where φ is the control function.

From now on we take $v_l(t)$ as given by Theorem 1.2. As part of Theorem 2 of [24] it is shown that for each $\epsilon > 0$ and each $t \geq 0$,

$$\mathbb{P}\left[|v_l^n(t) - v_l(t)| > \epsilon\right] \to 0 \tag{1.8}$$

as $n \to \infty$, providing that $v_l^n(0) \to v_l(0)$ and $\sum l^3 v_l(0) < \infty$. In fact, convergence was proven in a slightly stronger sense than (1.8) and we will show that convergence holds in a stronger sense still; in Section 2 we state the convergence theorem of [24] precisely and show that it can be upgraded into locally uniform convergence in probability, leading to the following result.

Theorem 1.5. *Suppose that, for each $l \in \mathbb{N}$, $v_l^n(0) \to v_l(0)$ as $n \to \infty$, where $\sum_l l^3 v_l(0) < \infty$. Then for each $\epsilon > 0$, and each $T > 0$,*

$$\mathbb{P}\left[\sup_{l \in \mathbb{N}} \sup_{s \in [0,T]} |v_l^n(s) - v_l(s)| > \epsilon\right] \to 0$$

as $n \to \infty$.

Recall that, for each l, the function $t \mapsto v_l(t)$ is continuous. So far, in keeping with [24], we have used left-continuous $v_l^n(t)$ (and left-continuous $\mathcal{Z}^n$). Note that Theorem 1.5 would also hold if we replaced v_l^n by its càdlàg version.

In fact, with Theorem 1.5 in hand it is advantageous to switch from working with left-continuous paths to working with càdlàg paths (i.e. right-continuous with left limits). Having càdlàg paths will be helpful to us because our main result (Theorem 1.7) is a result about convergence of jump processes and as part of its proof we will use standard results concerning martingales and stopping times.

To avoid unnecessary notation we will use the same symbols to refer to both versions; our convention is that up to this point and for the duration of Section 2 (in which Theorem 1.5 is proved) we use left-continuous paths but in all other sections (and for the remainder of Section 1) we use càdlàg paths.

1.2 Cluster Growth

The sequence $v_l^n(\cdot)$ characterizes the globally averaged behaviour of (the size of) all clusters present in $\mathcal{Z}^n$ as $n \to \infty$. Our aim in this paper is to paint a further level of detail into this picture by describing the behaviour of the cluster associated to a vertex chosen uniformly at random within $\mathcal{Z}^n$.

Let p be a vertex sampled uniformly at random from $[n]$ (independently of $\mathcal{Z}^n$) and set

$$\mathcal{C}_t^n = \mathcal{C}_t^n(p) \quad \text{and} \quad C_t^n = |\mathcal{C}_t^n|.$$

In order to understand the behaviour of C^n, let us consider heuristically the evolution when $C_t^n = k$. In this case the total rate of the growth clocks of edges with at least one endpoint in $\mathcal{C}_t^n(p)$ is $k(1 + \mathcal{O}(\frac{k}{n}))$ as $n \to \infty$, uniformly in k (see (4.5) for the exact rate). As $n \to \infty$ we typically have $n \gg k$ so, when the next new edge is connected to $\mathcal{C}_t^n(p)$, it is very unlikely for both the endpoints of this edge to be within $\mathcal{C}_t^n(p)$. Consequently the corresponding cluster $\mathcal{C}'$ to which $\mathcal{C}_t^n(p)$ connects looks very similar to a size biased sample of the clusters in $\mathcal{Z}_t^n$, that is $\mathbb{P}\left[|\mathcal{C}'| = j\right] \approx \mathbb{E}[v_j^n(t)]$.

In this paper we define and study C, a certain Markov branching process in a varying environment. We will show that C is the limit of the processes C^n as $n \to \infty$. In view of Theorem 1.5, if $t > T_{gel}$ and $C_t = k$, we expect C_t to increase at rate k to a size $k + L$ where L is a random variable whose distribution satisfies $\mathbb{P}[L \geq l] \asymp l^{-1/2}$. Such a process is explosive in finite time.

When C_t^n has size k it sees a fire at rate $k\lambda_n$, which tends to zero as $n \to \infty$. However, if C_t^n manages to grow large enough (in particular, to size $C_t^n \gg \frac{1}{\lambda_n}$) then the cluster $\mathcal{C}_t^n(p)$ will burn and C_t^n will return to 1. It is not obvious that C_t^n, started at size $k = \mathcal{O}(1)$, can grow to size $\frac{1}{\lambda_n}$ in $\mathcal{O}(1)$ time but in Section 4 we will show that in fact this does occur. Consequently, in the limit as $n \to \infty$ we expect to see an instantaneous return to 1 at each explosion time.

Let $E = \mathbb{N}$ and equip E with the topology such that $\lim_{n\to\infty} n = 1$ and 1 is the only non-isolated point of E. Note that E is compact and that the topology on E is metrizable, for example by the metric $d_E(i, j) = |f(i) - f(j)|$, where $f(i) = 1/i$ for $i \geq 2$ and $f(1) = 0$. We will use E as the state space for C, so that C is continuous at each of its explosion times.

We are now in a position to state our main result.

Definition 1.6. *Let $t \mapsto C_t$ be the unique càdlàg E-valued strongly Markov process such that:*

- *The distribution of C_0 is $k \mapsto v_k(0)$.*
- *C jumps out of state k with rate k. When such a jump occurs at (the random) time τ then, conditionally on τ, the value of C increases by L, sampled according to the distribution $\mathbb{P}_\tau [L = l] = v_l(\tau)$.*
- *C takes the value 1 at any accumulation point of jump times.*

The third condition implies that C_t moves continuously in E at the explosion times, so the only discontinuities in its path are the jumps described in the second condition.

Theorem 1.7. *Suppose $\sum l^3 v_l(0) < \infty$ and that $\lim_{n\to\infty} v_l^n(0) = v_l(0)$ for each l. Then there exists a coupling of C^n and C such that, for each $\epsilon > 0$ and $T > 0$,*

$$\mathbb{P}\left[\sup_{s\in[0,T]} d_E(C_s^n, C_s) > \epsilon\right] \to 0$$

as $n \to \infty$.

Remark 1.8. The coupling mentioned in Theorem 1.7 is constructed explicitly as part of the proof.

Note that Definition 1.6 provides a clear description of how the increments of C behave, but it does not offer a characterization of the distribution at fixed time. We rectify this with the following result, which will be proved as part of argument leading to Theorem 1.7.

Proposition 1.9. *Suppose that $\sum_l l^3 v_l(0) < \infty$. Then, for all $t \in [0, \infty)$ and all $l \in \mathbb{N}$, $\mathbb{P}[C_t = l] = v_l(t)$.*

Recall that the growth of the cluster of any fixed vertex in $\mathcal{Z}_t^n$ is driven by sampling increments from the (random) cluster size distribution of $\mathcal{Z}_t^n$, with a small modification to correct for the possibility that a new edge forms a cycle. In the limit as $n \to \infty$ the cluster size distribution becomes deterministic, so we expect the local limit of the cluster size of a fixed vertex to be strongly Markov (with respect to its generated filtration), even though C_t^n is not. In the finite model $\mathcal{Z}^n$, exchangeability implies that the distribution of the size of the cluster of a randomly sampled point is equal to the size biased distribution of the global distribution of cluster sizes. Proposition 1.9 shows that this property passes

meaningfully through the limit. The heuristic that we have just given for why Proposition 1.9 should hold true relies on Theorem 1.7, whereas in fact Proposition 1.9 will be a key step in our proof of Theorem 1.7.

1.3 Structure of the paper

In Section 1.4 we place the Erdős-Rényi forest fire model and our results in the context of some related models in the literature on coagulation-fragmentation processes.

In Section 2 we prove Theorem 1.5. This section refers to technical details of [24]. The main object of Section 3 is the proof of Proposition 1.9. This is done by analysing a linear control problem which characterizes the distribution of the process C_t. In Lemma 3.11 we provide a probabilistic interpretation of the associated characteristic curves that may be of independent interest. In Section 3.5 we establish the long-term average behaviour of φ and, as a consequence, we show that C_t explodes infinitely often. Finally, in Section 4 we prove Theorem 1.7 by constructing a coupling between the (finite) Erdős-Rényi forest fire model $\mathcal{Z}^n$ and the process C_t. An outline of this coupling is given at the start of Section 4.

Sections 2, 3 and 4 can be read essentially independently of one another. Section 3 does not rely on anything from Section 2, whilst Section 4 relies only on Sections 2 and 3 through the statements of Theorem 1.5 and Proposition 1.9.

Throughout Sections 2, 3 and 4, as well as Assumption 1.1 we assume without further comment the hypotheses on the initial conditions that appear in the statements of our main results, namely $\sum_l l^3 v_l(0) < \infty$ and that $v_l^n(0) \to v_l(0)$ as $n \to \infty$ for each $l \in \mathbb{N}$.

1.4 Relationships to other models

In general, long range interactions between large clusters are not easy to analyse rigorously, or even simulate. As a consequence, rigorous results concerning forest fire models are not common. One model in particular deserves special mention in comparison to our own. The Drossel-Schwabl model (introduced in [11]) differs from our own model in two important respects: its underlying graph is the lattice $[-n, n]^d$ and growth clocks correspond to vertices rather than edges. Despite receiving much attention in the physics literature, in the appropriate limit of the stationary two dimensional Drossel-Schwabl model, it is not even known whether the probability that the origin is occupied is less than or equal to 1 (as was noted by van den Berg and Brouwer [5], who investigate a closely related question).

Schenk et al. [25] gave a detailed non-rigorous description of the two dimensional Drossel-Schwabl model in its stationary state. They showed that in this case self-organized criticality occurs through the appearance of two qualitatively different types of fires, occurring simultaneously within the model but on different scales. Such multi-scale behaviour is often associated to self-organized criticality; see Preussner [22] for a detailed survey of the physics literature.

There is a natural connection between forest fire models and percolation, resting on the heuristic observation that taking a forest fire model and suppressing its fires results in a percolation model. As we saw in Section 1.1, in the Erdős-Rényi forest fire model this connection leads to the dynamical Erdős-Rényi model.

There has been recent interest in building a forest fire mechanism into percolation on $\mathbb{Z}^d$, by starting with supercritical percolation, burning the infinite cluster (but keeping the other finite clusters) and then asking what additional edge density must be added in order to create a new infinite cluster. This question was posed by van den Berg and Brouwer in [4] and was investigated for $d \geq 7$ by Ahlberg et al. [1] and for $d = 2$ by Kiss et al. [18].

The frozen percolation model, introduced by Aldous [2], is another hybrid of forest fires and percolation. In frozen percolation vertices in clusters that become infinite are instantly removed from the model and never return. Consequently, the total number of vertices in the model decreases as time passes; unlike (1.7) there is no influx of mass back into $v_1(t)$ and this makes the model somewhat easier to analyse. Frozen percolation is known to exhibit self-organized criticality and limit theorems concerning the size of the cluster of a typical vertex in a mean field model of frozen percolation have been established in Ráth [23]. There the freezing mechanism is similar to the lightning mechanism in our model, controlled by a rate λ_n that tends to 0 as the model size n tends to infinity. A forthcoming work of Martin and Ráth will give a precise description, in terms of the multiplicative coalescent, of the behaviour of the largest clusters in that model with $\lambda_n = n^{-1/3}$. Fournier and Laurençot [15] considered a similar model, the Marcus–Lushnikov process with a cut-off coagulation kernel, in which clusters are frozen when they exceed a certain threshold size $\alpha(n)$ which tends to infinity as $n \to \infty$. They showed convergence to either the Smoluchowski or Flory equation depending on the growth of the threshold function. The behaviour of this cut-off model after gelation has been studied in more detail by Merle and Normand [19].

Returning to forest fire models, attempts have been made to construct limits in the form of infinite interacting particle systems. In the case where the underlying graph is the integer lattice $[-n, n]^d$ and $\lambda_n = \lambda \in (0, \infty)$ stays constant as $n \to \infty$, it was shown in Dürre [12, 13, 14] that such a a limiting process exists. Stahl [26] showed that this process has a stationary distribution. Note that in this limit clusters will always burn while they are still of $\mathcal{O}(1)$ size.

In the case where the underlying graph is a regular tree, and with λ_n tending slowly, but not too slowly, to 0 (or rather, along a suitable subsequence where n is the number of vertices of a regular tree), Graf [16] has shown that the limit, up to and including the gelation time, is a dynamical version of self destructive percolation. Graf [17] considers the case where the underlying graph is the upper half plane of $\mathbb{Z}^2$ and, with a slightly different approximation scheme and burning mechanism, establishes tightness (but not uniqueness) of the limit.

One dimensional forest fire models have received much more rigorous treatment than dimensions greater than one; like our own model they have simplified spatial structure. Notably, Bressaud and Fournier [8] constructed a particle system limit of one dimensional forest fire models, in the appropriate scaling regime where λ tends to zero. In [9] the same authors find interesting limits of the equilibria of an infinite system of coupled ODEs, which was obtained from a one dimensional forest fire model by a mean field approximation. These equations are similar to the critical forest fire equations discussed in the present paper, but have a constant coalescence kernel instead of a multiplicative one, which is to say that large clusters wait as long to coalesce as small ones do, although clusters burn at a rate proportional to their size. Bertoin [7] investigates a forest fire version of Knuth's parking model, which is related to hashing with linear probing and, in a similar vein, van den Berg and Tóth [6] investigate a forest fire model related to signal processing and show that it exhibits self organized criticality. We refer the reader interested in the one dimensional case to the references therein.

We have already introduced the relationship between our model and the Smoluchowski coagulation equations with multiplicative kernel, in (1.2) and Theorem 1.2. A wide ranging survey of Smoluchowski coagulation equations and associated stochastic systems and be found in Aldous [3]. A derivation of Smoluchowski's equation as the limit in law of an appropriate (stochastic) particle system, along with existence and uniqueness results corresponding to quite general kernels can be found in [20, 21].

Deaconu et al. [10] study what, in our terminology, is the growth process of the

cluster of a tagged particle in the environment associated to Smoluchowski coagulation equations with more general kernels, up to time T_{gel}. In particular, they use an analogue of Proposition 1.9 to construct solutions to Smoluchowski coagulation equations over time $[0, T_{gel})$. By contrast, for (1.5)+(1.6) the conservation of mass beyond T_{gel} means that the tagged particle exhibits interesting behaviour after T_{gel} but existence and uniqueness of solutions is already known (see Theorem 1.2).

2 The Space Of Forest Fire Evolutions

Theorem 2 of Ráth and Tóth [24], which we seek to improve upon in this section, identifies the limit of the process $t \mapsto (v_l^n(t))_{l=1}^n$. In order to understand their result we must first describe the space in which $(v_l^n(t))_{l=1}^n$ lies.

Let $T > 0$. Let $\mathscr{W}_T$ be the space of paths $w : [0, T] \to [0, 1]$ that are left-continuous with right limits and are of bounded variation. Note that each such path $w(\cdot)$ can be written as

$$w(t) = w(0) + \mu_w[0, t) \tag{2.1}$$

for some finite signed measure μ_w on $[0, T]$. For $w^n, w \in \mathscr{W}_T$ we say that $w^n \to w$ if and only if $w^n(0) \to w(0)$ and $\mu_{w^n} \to \mu_w$ weakly as $n \to \infty$. This topology makes $\mathscr{W}_T$ a Polish space. Let

$$\mathscr{V} = \left\{ \mathbf{u} = (u_l)_{l=1}^\infty \,:\, u_l \geq 0 \text{ and } \sum_{l=1}^\infty u_l \leq 1 \right\}$$

and for each $T > 0$ let

$$\mathscr{E}_T = \{\mathbf{u} : [0, T] \to \mathscr{V} \,:\, \text{ for each } l, u_l(\cdot) \text{ is left-continuous and of bounded variation}\}$$

If $\mathbf{u}^n = (u_l^n(\cdot)) \in \mathscr{E}_T$ and $\mathbf{u} = (u_l(\cdot)) \in \mathscr{E}_T$ then we say

$$\mathbf{u}^n \to \mathbf{u} \quad \Longleftrightarrow \quad \text{for each } l,\ u_l^n \to u_l \text{ in } \mathscr{W}_T \tag{2.2}$$

where, again, the convergence on both sides is as $n \to \infty$. This topology makes $\mathscr{E}_T$ a Polish space. The space $\mathscr{E}_T$ is referred to in [24] as the space of 'forest fire evolutions' over the time interval $[0, T]$. (Note that in the definition of the topology of $\mathscr{E}_T$ in [24] the necessary condition of co-ordinatewise convergence of initial conditions is implicit although not explicitly stated).

We set $\mathbf{v}^n = (v_l^n(\cdot))$ and $\mathbf{v} = (v_l(\cdot))$. We consider these as elements of $\mathscr{E}_T$ (for each T) without comment by restricting the domains of the paths v_l^n and v_l to the time interval $[0, T]$.

Theorem 2.1 (Theorem 2, [24])**.** *Suppose that $\sum l^3 v_l(0) < \infty$ and that $v_l^n(0) \to v_l(0)$ for each $l \in \mathbb{N}$ as $n \to \infty$. Then, for each $T > 0$, $\mathbf{v}^n \to \mathbf{v}$ in probability in $\mathscr{E}_T$.*

We will now upgrade Theorem 2.1 into Theorem 1.5. In order to do this we will need to look a little way inside of the proof of Theorem 2.1 but first we record an elementary result.

Lemma 2.2. *Let $T > 0$ and let $w \in \mathscr{W}_T$ be continuous. For each $n \in \mathbb{N}$ let w^n be a $\mathscr{W}_T$ valued random variable such that the path w^n is increasing and suppose that $w^n \to w$ in probability in $\mathscr{W}_T$. Then, for each $\epsilon > 0$, $\mathbb{P}[\sup_{s \in [0,T]} |w^n(s) - w(s)| > \epsilon] \to 0$ as $n \to \infty$.*

Proof. We have $w^n \to w$ in probability and, since $\mathscr{E}_T$ is separable, we may apply the Skorohod Representation Theorem and assume that $w^n \to w$ almost surely (after a change of our underlying probability space). Thus $\mu_{w^n} \to \mu_w$ almost surely, in the sense of the weak topology on measures on $[0, T]$, and $w^n(0) \to w(0)$ almost surely. Since w is continuous the signed measure μ_w defined by (2.1) is non-atomic. Hence

$w(t) = w(0) + \mu_w[0,t]$ for any $t \in [0,T]$. Since μ_{w^n} is a non-negative measure for all n, μ_w is also a non-negative measure. Further, $\mu_w\{0,t\} = 0$ so $[0,t]$ is a μ_w-continuity set. The Portmanteau Theorem thus implies that $\mu_{w^n}[0,t] \to \mu_w[0,t]$ almost surely. Similarly, $\mu_{w^n}\{t\} \to \mu_w\{t\} = 0$ almost surely so we can conclude that almost surely

$$w^n(t) = w^n(0) + \mu_{w^n}[0,t] - \mu_{w^n}\{t\} \to w(0) + \mu_w[0,t] = w(t)\,. \tag{2.3}$$

Now let $\epsilon > 0$. Since $[0,T]$ is compact, w is uniformly continuous and hence there exists $\delta > 0$ such that for all $|s-t| < \delta$, $|w(t) - w(s)| < \epsilon$. We may take $\delta = T/M$ for some positive integer M. The set $\mathcal{T} = \{0, \delta, 2\delta, \ldots, (M-1)\delta, T\}$ is finite, hence

$$\mathbb{P}\left[\sup_{t\in\mathcal{T}} |w^n(t) - w(t)| > \epsilon\right] \to 0 \text{ as } n \to \infty. \tag{2.4}$$

Fix $s \in [0,T]$. Then there is some integer $0 \le k < M$ such that $k\delta \le s \le (k+1)\delta$. Since w^n is increasing we have $w^n(k\delta) \le w^n(s) \le w^n((k+1)\delta)$ and from the uniform continuity of w we have

$$|w(s) - w(k\delta)| \le \epsilon, \quad |w(s) - w((k+1)\delta)| \le \epsilon.$$

On the complement of the event in (2.4) we have also that

$$|w^n(k\delta) - w(k\delta)| \le \epsilon, \quad |w^n((k+1)\delta) - w((k+1)\delta)| \le \epsilon$$

and we are thus able to conclude that $\mathbb{P}[\sup_{s\in[0,T]} |w^n(s) - w(s)| \ge 2\epsilon] \to 0$ as $n \to \infty$, which completes the proof. □

We now describe the evolution of v_k^n in terms of the two forces affecting it: coagulating clusters and burning clusters. For $t \in [0,T]$ let $Q_{j,k}^n(t)$ be the number of clusters of size j in $\mathcal{Z}^n$ that are lost during the interval $[0,t]$ due to coagulations with clusters of size k. Thus for $j \ne k$, $Q_{j,k}^n(t)$ is the number of times during $[0,t]$ that a cluster of size k and a cluster of size j coagulate to form a cluster of size $j+k$, within $\mathcal{Z}^n$, and $Q_{j,k}^n(t) = Q_{k,j}^n(t)$. Note that $Q_{j,j}^n(t)$ is *twice* the number of times during $[0,t]$ that two clusters of size j coagulate to form a cluster of size $2j$. For $j > 1$ define R_j^n to be j times the number of times during $[0,t]$ that a cluster of size j burns. Then set

$$q_{j,k}^n(t) = \frac{Q_{j,k}^n(t)}{n}, \quad q_k^n(t) = \sum_{l=1}^{\infty} q_{k,l}(t), \quad r_j^n(t) = \frac{R_j^n(t)}{n}, \quad r^n(t) = \sum_{k=2}^{\infty} r_k^n(t).$$

It is readily seen from the definition of $\mathcal{Z}^n$ that

$$v_k^n(t) = v_k^n(0) + \frac{k}{2}\sum_{l=1}^{k-1} q_{l,k-l}^n(t) - kq_k^n(t) - r_k^n(t) + \mathbb{1}\{k=1\}r^n(t). \tag{2.5}$$

We now collate results from Proposition 1, equations (19), (37) and Theorem 2 of [24]. There are continuous functions $q_{j,k}(\cdot)$, $q_k(\cdot)$, and $r_k(\cdot)$ and $r(\cdot)$ such that for all j,k

$$q_{j,k}^n \to q_{j,k}, \quad q_k^n \to q_k, \quad r_k^n \to r_k, \quad r^n \to r \tag{2.6}$$

in $\mathscr{W}_T$ for any $T > 0$ and, further,

$$v_k(t) = v_k(0) + \frac{k}{2}\sum_{l=1}^{k-1} q_{l,k-l}(t) - kq_k(t) - r_k(t) + \mathbb{1}\{k=1\}r(t)$$

where $q_k(t) = \sum_{l=1}^{\infty} q_{k,l}(t)$. In fact $r_k(\cdot) = 0$ (heuristically, this is because a cluster of size k burns at rate $k\lambda_n \to 0$) but we will continue to write r_k for symmetry. Finally, $r(t) = 0$ for $0 \le t \le T_{gel}$ but $r(t) > 0$ and r(t) is strictly increasing for $t > T_{gel}$. In particular for $t > T_{gel}$ we have $r(t) \ne \sum_{k=2}^{\infty} r_k(t)$.

Proof of Theorem 1.5. Let $\epsilon > 0$ and $T \in (0, \infty)$. We will prove the theorem in two steps, the first of which is to show that for each $k \in \mathbb{N}$,

$$\mathbb{P}\left[\sup_{s\in[0,T]} |v_k^n(t) - v_k(t)| > \epsilon\right] \to 0. \tag{2.7}$$

as $n \to \infty$.

Let us first look at $k \geq 2$. In this case we can write

$$v_k^n(t) - v_k(t) = v_k^n(0) - v_k(0) + \overbrace{\left(\frac{k}{2}\sum_{l=1}^{k-1} q_{l,k-l}^n(t) + kq_k(t) + r_k(t)\right)}^{f_1(t)} - \overbrace{\left(\frac{k}{2}\sum_{l=1}^{k-1} q_{l,k-l}(t) + kq_k^n(t) + r_k^n(t)\right)}^{f_2(t)}$$

We have that $v_k^n(0) - v_k(0)$ converges to zero as $n \to \infty$. Moreover, both f_1 and f_2 are increasing functions and elements of $\mathscr{W}_T$. Equation (2.6) implies that f_1 and f_2 both converge (in $\mathscr{W}_T$) to

$$\frac{k}{2}\sum_{l=1}^{k-1} q_{l,k-l}(t) + kq_k(t) + r_k(t).$$

Applying Lemma 2.2 to f_1 and f_2 respectively, we obtain that in fact $v_k^n - v_k$ tends to 0 locally uniformly in probability, which proves (2.7) for $k \geq 2$.

The case $k = 1$ remains. In this case, $v_1^n(t) - v_1(t)$ has the additional term $r^n(t) - r(t)$. From (2.6) we have $r^n \to r$ in $\mathscr{W}_T$ and using Lemma 2.2 we deduce that $r^n \to r$ locally uniformly in probability. Combining this fact with the argument used in the $k \geq 2$ case, we have proved the $k = 1$ case of (2.7) and thus completed the proof of (2.7).

We now deduce Theorem 1.5 from (2.7). By Theorem 1.2, for each k, $t \mapsto v_k(t)$ is continuous on $[0, T]$. By Dini's theorem, we can choose $K \in \mathbb{N}$ such that

$$\sup_{s\in[0,T]} \sum_{k=K+1}^{\infty} v_k(s) \leq \frac{\epsilon}{3}. \tag{2.8}$$

Hence also $\sup_{s\in[0,T]} \sup_{k>K} v_k(s) \leq \frac{\epsilon}{3}$. Using (2.7), let $N \in \mathbb{N}$ be such that for all $n \geq N$,

$$\mathbb{P}\left[\exists k \leq K, \sup_{s\in[0,T]} |v_k^n(t) - v_k(t)| \geq \frac{\epsilon}{3K}\right] \leq \epsilon. \tag{2.9}$$

Using (1.1) and (1.6), we note that, for $k > K$,

$$\begin{aligned}
\sup_{s\in[0,T]} v_k^n(s) &\leq \sup_{s\in[0,T]} \sum_{l=K+1}^{\infty} v_l^n(s) \\
&= \sup_{s\in[0,T]} \left(1 - \sum_{l=1}^{K} v_l(s) - \sum_{k=1}^{K} (v_l^n(s) - v_l(s))\right) \\
&= \sup_{s\in[0,T]} \left(\sum_{l=K+1}^{\infty} v_l(s) - \sum_{l=1}^{K} (v_l^n(s) - v_l(s))\right) \\
&\leq \frac{\epsilon}{3} + \sum_{l=1}^{K} \sup_{s\in[0,T]} |v_l^n(s) - v_l(s)|
\end{aligned} \tag{2.10}$$

Note that to obtain the last line of the above we used (2.8). Note also that the final line is independent of k. Using(2.10), followed by another application of (2.8) and then two applications of (2.9), we have

$$\begin{aligned}
&\mathbb{P}\left[\sup_{k\in\mathbb{N}}\sup_{s\in[0,T]}|v_k^n(s)-v_k(s)|>\epsilon\right]\\
&\qquad\le \mathbb{P}\left[\sup_{k\le K}\sup_{s\in[0,T]}|v_k^n(s)-v_k(s)|>\epsilon\right]+\mathbb{P}\left[\sup_{k>K}\sup_{s\in[0,T]}v_k^n(s)+v_k(s)>\epsilon\right]\\
&\qquad\le \mathbb{P}\left[\sum_{k=1}^{K}\sup_{s\in[0,T]}|v_k^n(s)-v_k(s)|>\frac{\epsilon}{3}\right]\\
&\qquad\qquad\qquad+\mathbb{P}\left[\frac{\epsilon}{3}+\sum_{k=1}^{K}\sup_{s\in[0,T]}|v_k^n(s)-v_k(s)|+\sup_{k>K}\sup_{s\in[0,T]}v_k(s)\ge\epsilon\right]\\
&\qquad\le \mathbb{P}\left[\sum_{k=1}^{K}\sup_{s\in[0,T]}|v_k^n(s)-v_k(s)|>\frac{\epsilon}{3}\right]+\mathbb{P}\left[\sum_{k=1}^{K}\sup_{s\in[0,T]}|v_k^n(s)-v_k(s)|\ge\frac{\epsilon}{3}\right]\\
&\qquad\le 2\epsilon.
\end{aligned}$$

This completes the proof of Theorem 1.5. □

3 Cluster Growth in the Limiting Process

In this section we investigate the process C which, in the next section, will be shown to be the limit of (C^n). Recall that, from this point on, we use càdlàg versions of all processes. The main goal of this section is to prove Proposition 1.9, which states that $\mathbb{P}[C_t=\ell]=v_\ell(t)$ for all $t>0$. Our strategy for the proof is as follows. Recall that in Theorem 1.2 we gave a system of ODEs for the evolution of the (v_k); naturally they can also be expressed as a system of integral equations. We define

$$u_k(t)=\mathbb{P}\left[C_t=k\right]$$

and try set up a similar system of integral equations for the (u_k). We then have a description of the evolution of u_k-v_k and seek to show that in fact u_k-v_k is identically zero.

It is convenient to use the probability generating functions

$$X_t(z)=\sum_{k=1}^{\infty}z^k v_k(t),\qquad Y_t(z)=\sum_{k=1}^{\infty}z^k u_k(t)=\mathbb{E}\left(z^{C_t}\right).$$

Both X_t and Y_t are power series in z with non-negative coefficients summing to 1, which therefore converge uniformly on the closed unit disc in the complex plane. Thus they define analytic functions on the open unit disc $\mathbb{D}$ with continuous extension to the closed unit disc $\overline{\mathbb{D}}$, and we have $X_t(1)=Y_t(1)=1$. We will mostly be concerned with the behaviour of $X_t(z)$ and $Y_t(z)$ for $z\in[0,1]$.

We define

$$Z_t(z)=Y_t(z)-X_t(z).$$

Since $\mathbb{P}(C_0=k)=v_k(0)$ we have $Z_0(\cdot)=0$. We seek to show that Z_t is identically zero for all $t\ge 0$ by integrating along characteristic curves.

3.1 Properties of the environmental generating function

Let us fix some notation for partial derivatives. Given a function $f(\cdot,\cdot)$ or $f_\cdot(\cdot)$ of two variables, where the first or subscripted variable is a time coordinate and the other

variable is spatial, i.e. the variable of a generating function, we will write $\dot{f}$ for the partial derivative of f with respect to the time coordinate and f' for the partial derivative with respect to the spatial coordinate. In the case of functions of two variables that are both time coordinates we will not use the dot notation but will write the derivatives explicitly.

The generating function analysis in Ráth and Tóth [24] uses the modified moment generating function

$$V(t,x) = -1 + \sum_{k=1}^{\infty} v_k(t) e^{-kx},$$

which was shown to be a solution to the *critical Burgers control problem*

$$\dot{V}(t,x) = -V'(t,x)V(t,x) + e^{-x}\varphi(t) \tag{3.1}$$

subject to the boundary conditions $V(t,0) = 0$ and $V(0,x) = V_0(x)$. The function φ is known as the *control function*. Recall that φ appeared in the statement of Theorem 2.1; $\varphi(t)$ is the infinitesimal rate at time t at which mass burns and returns to state 1. The moment generating function $V(t,x)$ is related to our probability generating function $X_t(z)$ by

$$V(t,x) = -1 + X_t(e^{-x}), \tag{3.2}$$

and thus (3.1) is equivalent to

$$\dot{X}_t(z) = zX_t'(z)(X_t(z) - 1) + z\varphi(t). \tag{3.3}$$

Using equation (3.2), equations (126) and (127) of [24] translate into estimates about the singularity of the probability generating function $X_t(z)$ at $z = 1$. In particular for any $w_0 \in (0,1]$ and $\bar{t} > T_{gel}$, uniformly on $(t,w) \in [T_{gel}, \bar{t}] \times (w_0, 1]$ we have

$$1 - X_t(1 - w^2) = \sqrt{2\varphi(t)}\, w(1 + O(1-w)) \tag{3.4}$$

and

$$X_t'(1 - w^2) = \sqrt{\frac{\varphi(t)}{2}}\, w^{-1}(1 + O(1-w)). \tag{3.5}$$

In fact these estimates hold on $[T_{gel}, \bar{t}] \times (0,1]$ as a consequence of the following elementary lemma.

Lemma 3.1. *Let $f(z) = \sum_{k=1}^{\infty} a_k z^k$ be a power series where $\sum_k |a_k| = B < \infty$. Then f defines a continuous function on $\overline{\mathbb{D}}$ with an analytic restriction to $\mathbb{D}$. For all $z \in \mathbb{D}$ we have*

$$|f(z)| \le \sum_k |a_k||z| \le B|z|$$

and for each $n \ge 1$,

$$|f^{(n)}(z)| \le \sum_{k=1}^{\infty} B \left| \frac{d^n}{dz^n} z^k \right| \;=\; \frac{n!\, B}{(1-|z|)^{n+1}}\,.$$

In particular each derivative of a probability generating function is locally bounded on $\mathbb{D}$ and the bound does not depend on the probability distribution.

Lemma 3.2. *$X_t(z)$ is continuous as a function of $(t,z) \in [0,\infty) \times \overline{\mathbb{D}}$. Moreover, $X_t(z)$ is continuously differentiable as a function of $(t,z) \in ([0, T_{gel}) \cup (T_{gel}, \infty)) \times \mathbb{D}$.*

Proof. According to Theorem 1.2, $v_k(t)$ is a continuous function of t for each $k \ge 1$. Since $v_k(t) \ge 0$ and $\sum_{k=1}^{\infty} v_k(t) = 1$ for each t, Dini's theorem implies that $t \mapsto (v_k(t))_{k=1}^{\infty}$ is continuous as a map from $[0,\infty]$ to ℓ^1. By Lemma 3.1, $|X_t(z) - X_s(z)| \le \|v(t) - v(s)\|_1$ for

all $z \in \overline{\mathbb{D}}$, so $X_t(z)$ is continuous in t, uniformly in z, and continuous in z for each t. It follows that $X_t(z)$ is jointly continuous as required.

Lemma 3.1 shows that $X_t'(z)$ is continuous in $z \in \mathbb{D}$, uniformly in t. For each fixed $z \in \mathbb{D}$, the power series $X_t'(z) = \sum_{k=1}^{\infty} k v_k(t) z^k$ is a uniform limit of continuous functions of t, therefore continuous in t for each fixed $z \in \mathbb{D}$. Hence $X_t'(z)$ is continuous on $[0,\infty) \times \mathbb{D}$.

To see that $\dot{X}_t(z)$ is continuous on $([0, T_{gel}) \cup (T_{gel}, \infty)) \times \mathbb{D}$, consider the right-hand side of equation (3.3). Since φ vanishes on $[0, T_{gel})$ and is continuous on $[T_{gel}, \infty)$, both summands are jointly continuous in t and z in the given domain. Since both partial derivatives are continuous, we conclude that $X_t(z)$ is continuously differentiable on the same domain. We remark that although $\dot{X}_t(z)$ has a jump at $t = T_{gel}$ when $z \neq 0$, it has left and right limits that depend continuously on z. □

3.2 Characteristic curves for the Flory equation

Recall that the Erdős-Rényi case $\lambda_n = 0$ is described in the limit by the Flory coagulation equations with multiplicative kernel:

$$\dot{s}_k(t) = -k s_k(t) + \sum_{l=1}^{k-1} l s_l(t) s_{k-l}(t) \quad \text{for } k \geq 1. \tag{3.6}$$

Note the rearrangement $\frac{k}{2} \sum_{l=1}^{k-1} s_l(t) s_{k-l}(t) = \sum_{l=1}^{k-1} l s_l(t) s_{k-l}(t)$. Even with general initial conditions the equations (3.6) can be solved inductively, starting with $k = 1$, and it is easy to see that they have a unique solution. As a warm-up for the analysis later in this section, we describe how the solution can be given using generating functions and characteristic curves. The results in this section are not new, but it is useful to have them in our own terminology.

The infinite system of ODEs (3.6) is equivalent to the PDE

$$\dot{S}_t(z) = z S_t'(z)(S_t(z) - 1)\,, \tag{3.7}$$

for the probability generating function $S_t(z) = \sum_{k=1}^{\infty} z^k s_k(t)$. Note that (3.7) is precisely (3.3) without control term $z\varphi(t)$. For any $0 \leq w < 1$, define $\psi_w(t)$ for $t \geq 0$ by

$$\psi_w(t) = w e^{t(1 - S_0(w))}.$$

In particular $\psi_w(0) = w$. Then $\frac{d}{dt}\psi_w(t) = (1 - S_0(w))\psi_w(t)$ and while $\psi_w(t) < 1$ we have

$$\begin{aligned}
\frac{d}{dt} S_t(\psi_w(t)) &= \dot{S}_t(\psi_w(t)) + S_t'\left(\psi_w(t)\right)\left(\frac{d}{dt}\psi_w(t)\right) \\
&= \psi_w(t) S_t'\left(\psi_w(t)\right)\left(S_t\left(\psi_w(t)\right) - 1\right) + S_t'\left(\psi_w(t)\right)\left(1 - S_0\left(\psi_w(0)\right)\right)\psi_w(t) \\
&= \psi_w(t) S_t'(\psi_w(t))\left(S_t\left(\psi_w(t)\right) - S_0\left(\psi_w(0)\right)\right)
\end{aligned}$$

Grönwall's inequality shows that the unique solution of the above equation is $S_t(\psi_w(t)) = S_0(\psi_w(0))$, so in fact ψ_w is a characteristic curve of (3.7) and satisfies

$$\frac{d}{dt}\psi_w(t) = (1 - S_t(\psi_w(t)))\psi_w(t),$$

as long as $\psi_w(t) < 1$. Hence, to find $S_t(z)$ for some $t > 0$ and $z \in (0,1)$, we must find a value of w for which $\psi_w(t) = z$, and then $S_t(z) = S_t\left(\psi_w(t)\right) = S_0\left(\psi_w(0)\right) = S_0(w)$. Thus we have to solve the implicit equation

$$\log w + t(1 - S_0(w)) = \log z\,. \tag{3.8}$$

For each fixed value of t, the function $\log w + t(1 - S_0(w))$ on the left-hand side of (3.8) is a concave function of w which tends to $-\infty$ as $w \searrow 0$ and tends to 0 as $w \nearrow 1$. Since $\log z < 0$, there is a unique choice of $w \in (0,1)$ such that $\psi_w(t) = z$.

Lemma 3.3. *Define* $S_0'(1-) = \lim_{z\nearrow 1} S_0'(z) = \sum_{k=1}^{\infty} k s_k(0)$. *Then* $T_{gel} = 1/S_0'(1-)$ *is the* gelation time*: up to* T_{gel} *the solution of* (3.6) *is* conservative*, meaning that* $\sum_{k=1}^{\infty} s_k(t) = 1$*, but after* T_{gel} *we have* $\sum_{k=1}^{\infty} s_k(t) < 1$*, indicating that mass has been lost into the gel (or giant component).*

Proof. Gelation is encoded by the condition $\sum_{k=1}^{\infty} s_k(t) = S_t(1) = \lim_{z\nearrow 1} S_t(z) < 1$. We have

$$\frac{\partial}{\partial w}\psi_w(t) = (1 - twS_0'(w))\, e^{t(1-S_0(w))}$$

so the mapping $w \mapsto \psi_w(t)$ is strictly increasing on $[0,1]$ if $t < 1/wS_0'(w)$ for all $w \in (0,1)$. This holds when $t \le 1/S_0'(1-) = T_{gel}$. Since $\psi_0(t) = 0$ and $\lim_{w\nearrow 1} \psi_w(t) = 1$ for all $t \ge 0$, the mapping $w \mapsto \psi_w(t)$ is a homeomorphism of $(0,1)$ onto itself when $t \le T_{gel}$, hence

$$S_t(1) = \lim_{z\nearrow 1} S_t(z) = \lim_{w\nearrow 1} S_0(w) = 1\,.$$

For each fixed value of w the characteristic curve $\psi_w(\cdot)$ begins at $\psi_w(0) = w$ and increases to reach the value $\psi_w(t) = 1$ at $t = -\log w / (1 - S_0(w))$, beyond which $S_t(\psi_w(t))$ is no longer defined. For $t > T_{gel}$ we claim there is a unique $w_t < 1$ such that

$$\log w_t + t\,(1 - S_0(w_t)) \;=\; 0\,, \quad \text{i.e. } \psi_{w_t}(t) = 1.$$

To see this note that for w sufficiently close to 1,

$$\frac{d}{dw}\left(\log w + t\,(1 - S_0(w))\right) = \frac{1}{w} - tS_0'(w) < 0\,,$$

so the concave function $\log w + t\,(1 - S_0(w))$ has a positive maximum on $(0,1)$ and a root between 0 and the value at which this maximum is achieved. A short calculation shows that for $0 < w < 1$,

$$-(\log w)wS_0'(w) + S_0(w) < \lim_{w\nearrow 1}\left(-(\log w)wS_0'(w) + S_0(w)\right) \;=\; 1$$

because the left-hand side of this inequality is increasing in w. Hence when $\psi_w(t) < 1$ we have

$$wS_0'(w) < \frac{1 - S_0(w)}{-\log w} < \frac{1}{t} \quad \text{and} \quad \frac{\partial}{\partial w}\psi_w(t) > 0\,.$$

Therefore the mapping $w \mapsto \psi_w(t)$ is an increasing homeomorphism from $(0, w_t)$ onto $(0,1)$, and

$$S_t(1) = \lim_{z\nearrow 1} S_t(z) = \lim_{w\nearrow w_t} S_0(w) = S_0(w_t) < 1\,.$$

□

Lemma 3.4. *For all* $0 \le t < T_{gel}$*, the limiting mean cluster size is given by*

$$\sum_{k=1}^{\infty} k v_k(t) = (T_{gel} - t)^{-1}.$$

Proof. It was remarked above that if $S_0 = V_0$ then the solution of equations (1.6) and (1.5) coincides with the solution of the Flory coagulation equations (1.2) up to the time $T_{gel} = \left(\sum_{k=1}^{\infty} k v_k(0)\right)^{-1}$ and no later. Define

$$x_k(t) = \sum_{\ell=k+1}^{\infty} v_\ell(t), \qquad E(t) = \sum_{k=1}^{\infty} k v_k(t) = \sum_{k=0}^{\infty} x_k(t).$$

We may assume that $E(0) < \infty$ or equivalently $T_{gel} > 0$, since otherwise there is nothing to prove. Using (1.5) and (1.6) we find that $x_0(t) = 1$ and for all $k \geq 1$

$$\dot{x}_k(t) = \sum_{\ell=1}^{k} \ell v_\ell(t) x_{k-\ell}(t) - \varphi(t)\,.$$

It follows that for $t < T_{gel}$, when $\varphi(t) = 0$, we have $\dot{x}_k(t) \geq 0$ for all $k \geq 0$, so $E(t)$ is increasing. A convergent series of increasing functions may be differentiated term-by-term and doing so shows that $\frac{dE(t)}{dt} = E(t)^2$. Given the initial condition $E(0) = \sum_k k v_k(0) = T_{gel}^{-1}$, this has the unique solution $E(t) = (T_{gel} - t)^{-1}$. □

3.3 Characteristic curves for the critical forest fire equations

We now move on to define characteristic curves for equation (3.3).

Lemma 3.5. *For each $y > T_{gel}$ there exists a unique continuous function $\psi_y : [0, \infty) \to (0, 1]$ such that $\psi_y(t) = 1$ for all $t \geq y$, $\psi_y(t) < 1$ for all $t < y$, and*

$$\frac{d\psi_y(t)}{dt} = \psi_y(t)\big(1 - X_t(\psi_y(t))\big). \tag{3.9}$$

$\psi_y(t)$ is increasing and continuously differentiable on $(0, \infty)$. The function $y \mapsto \psi_y(0)$ is continuous and strictly decreasing, mapping (T_{gel}, ∞) onto $(\gamma, 1)$ for some $\gamma \in [0, 1)$.

Remark 3.6. We will construct the solution to (3.9) by working backwards from time y to time T_{gel} and then from T_{gel} to 0. It is convenient to deal separately with the time intervals $[0, T_{gel}]$ and $[T_{gel}, y]$ because $X_t(\cdot)$ has an algebraic singularity at 1, described by (3.4) and (3.5), when $t \geq T_{gel}$, while $X_t(\cdot)$ has no singularity at 1 when $t < T_{gel}$. The result of this is that distinct characteristic curves can coalesce at the value 1, but coalescence occurs only after time T_{gel}, since the initial value problem given by (3.9) with initial condition $\psi_y(0)$ has a unique solution up to time y.

Remark 3.7. The utility of the characteristic curve ψ_y lies in the fact that

$$\begin{aligned}\frac{d}{dt}\left(X_t\left(\psi_y(t)\right)\right) &= X_t'(\psi_y(t))\frac{d\psi_y(t)}{dt} + \dot{X}_t\left(\psi_y(t)\right)\\ &= \psi_y(t)\big(X_t'(\psi_y(t))(1 - X_t(\psi_y(t)) + X_t'(\psi_y(t))(X_t(\psi_y(t)) - 1) + \varphi(t)\big)\\ &= \psi_y(t)\varphi(t).\end{aligned} \tag{3.10}$$

In particular, on $[0, T_{gel})$ where $\varphi \equiv 0$ we see that $X_t(\psi_y(t))$ is constant, so

$$X_t(\psi_y(t)) = X_0(\psi_y(0))$$

and $\frac{d}{dt}\psi_y(t) = \psi_y(t)\left(1 - X_0\left(\psi_y(0)\right)\right)$, which implies that $\psi_y(t) = \psi_y(0)e^{t(1 - X_0(\psi_y(0)))}$.

Remark 3.8. We will show in Lemma 3.16 that $\gamma = 0$, which is to say that $[0, \infty) \times (0, 1)$ is filled by characteristic curves, but the proof will rely on Proposition 1.9.

Proof of Lemma 3.5. Let $y > T_{gel}$. The characteristic curve $\psi_y(t)$ is defined for $t \geq y$ by $\psi_y(t) = 1$. Our first task is to extend this solution continuously to $[T_{gel}, \infty)$ so that $\psi_y(t) < 1$ for $T_{gel} \leq t < y$. To do this we will make a change of variable to remove the singularity, and apply Picard's theorem.

We aim to express $\psi_y(t)$ in the form $\psi_y(t) = 1 - \upsilon_y(t)^2$, where $\upsilon_y(t) : [T_{gel}, y] \to [0, 1)$ is continuous and strictly decreasing, satisfies $\upsilon_y(y) = 0$, and for $t \in (T_{gel}, y)$ satisfies

$$\frac{d}{dt}\upsilon_y(t) = \frac{1}{2}(\upsilon_y(t)^2 - 1)\,\frac{(1 - X_t(1 - \upsilon_y(t))^2)}{\upsilon_y(t)}\,.$$

The point of this change of variable is to enable us to avoid constructing the constant solution $\psi_y(\cdot) = 1$ of (3.9), which would correspond to $\upsilon_y(\cdot) = 0$. Note that this solution would violate our condition that $\psi_y(t) < 1$ for $t < y$. To see how the change of variable helps, note that from equation (3.4) we have

$$\frac{1 - X_t(1-w^2)}{w} = \sqrt{2\varphi(t)}(1 + \mathcal{O}(w))$$

uniformly on $[T_{gel}, y] \times (0, 1]$. Hence, the function $F : [T_{gel}, y] \times \mathbb{R} \to \mathbb{R}$ given by

$$F(t,w) = \begin{cases} 1 & \text{for } w > 1 \\ \frac{1-X_t(1-w^2)}{w} & \text{for } w \in (0,1] \\ \sqrt{2\varphi(t)} & \text{for } w \le 0 \end{cases}$$

is continuous and strictly positive on $[T_{gel}, y] \times \mathbb{R}$. In fact, using (3.5) as well as (3.4) we have that uniformly on $(t, w) \in [T_{gel}, y] \times (0, 1]$,

$$\begin{aligned} \frac{\partial F(t,w)}{\partial w} &= 2X_t'(1-w^2) - w^{-2}(1 - X_t(1-w^2)) \\ &= 2\sqrt{\frac{\varphi(t)}{2}}w^{-1}\big(1 + \mathcal{O}(w)\big) - \sqrt{2\varphi(t)}w^{-1}\big(1 + \mathcal{O}(w)\big) \\ &= \mathcal{O}(1) \end{aligned}$$

It follows that, within $[T_{gel}, y] \times \mathbb{R}$, the function $F(t, w)$ is continuous with respect to t and Lipschitz with respect to w, uniformly in t. Therefore, it follows from Picard's Theorem that there is a unique solution $\upsilon_y : [T_{gel}, y] \to \mathbb{R}$ of the equation

$$\upsilon_y(t) = \int_t^y \frac{1}{2}(1 - \upsilon_y(s)^2)F(s, \upsilon_y(s))\, ds, \qquad \upsilon_y(y) = 0 \tag{3.11}$$

and, further, that $\upsilon_y(\cdot)$ is continuous.

We aim now to show that $0 < \upsilon_y(t) < 1$ for all $t \in [T_{gel}, y)$, and that υ_y is strictly decreasing there. From (3.11) we obtain

$$\frac{d}{dt}\upsilon_y(t) = \frac{1}{2}(\upsilon_y(t)^2 - 1)F(t, \upsilon_y(t))\,, \tag{3.12}$$

and hence υ_y is also the (unique) solution of the equation

$$f(t) = \upsilon_y(t_0) + \int_{t_0}^t \frac{1}{2}(f(s)^2 - 1)F(t, f(s))\, ds\,, \tag{3.13}$$

for any $t_0 \in [T_{gel}, y]$.

Since $(w^2 - 1)F(t, w) \ge 0$ when $w \ge 1$ it follows that $\upsilon_y(t) < 1$ for all $t \in [T_{gel}, y]$, for if this were to fail at $t = t_0$ then (3.13) shows that υ_y would be increasing on $[t_0, y]$, contradicting $\upsilon_y(y) = 0$.

We now show that $\upsilon_y(t) > 0$ for $t \in [T_{gel}, y)$. Suppose that $\upsilon_y(t) \le 0$ for some $t < y$. Then consider

$$t_0 = \sup\{t \in [T_{gel}, y) \,:\, \upsilon_y(t) \le 0\}\,.$$

We must have $t_0 < y$, since $\frac{d}{dt}\upsilon_y(t) \mid_{t=y} < 0$. Since υ_y is continuously differentiable by 3.12, we must have $\upsilon_y(t_0) = 0$, and $\frac{d}{dt}\upsilon_y(t_0) \ge 0$ which contradicts (3.12). Hence $0 < \upsilon_y(t) < 1$ for $T_{gel} \le t < y$, the integrand on the right-hand side of (3.11) is strictly positive, and $t \mapsto \upsilon_y(t)$ is strictly decreasing.

Since $0 < \upsilon_y(t) < 1$ for all $t \in [T_{gel}, y)$, from the definition of F we have

$$\frac{d\upsilon_y(t)}{dt} = \frac{1}{2}(\upsilon_y(t)^2 - 1)\left(\frac{1 - X_t(1 - \upsilon_y(t)^2)}{\upsilon_y(t)}\right).$$

We now define

$$\psi_y(t) = 1 - \upsilon_y(t)^2$$

and it follows that ψ_y has the desired properties: it is continuous and strictly increasing on $[T_{gel}, y]$ and satisfies (3.9) there, $\psi_y(y) = 1$, and $0 < \psi_y(t) < 1$ for $t \in [T_{gel}, y)$. Note that $\psi_y(t) \to 1$ as $t \nearrow y$ and

$$\frac{d}{dt}\psi_y(t) = -2\upsilon_y(t)\frac{d}{dt}\upsilon_y(t) \to 0 \quad \text{as } t \nearrow y.$$

Having constructed $\psi_y(t)$ on the interval $[T_{gel}, y]$, we can extend the solution uniquely back from time T_{gel} to time $t = 0$, using Picard's theorem applied directly to equation (3.9). We use $\Psi = \psi_y(T_{gel})$, which has already been defined above, as our 'initial' condition.

To this end, for $t \in [0, T_{gel}]$ we define

$$G(t,z) = \begin{cases} \psi_y(T_{gel})(1 - X_t(\psi_y(T_{gel}))) & \text{for } z > \psi_y(T_{gel}) \\ z(1 - X_t(z)) & \text{for } z \in [0, \psi_y(T_{gel})] \\ 0 & \text{for } z < 0. \end{cases}$$

Then $G(t,z)$ is continuous on $[0, T_{gel}] \times \mathbb{R}$ and for $z \in (0, \psi_y(T_{gel})$ and $t \in [0, T_{gel}]$ we have

$$\frac{\partial G(t,z)}{\partial z} = 1 - X_t(z) - zX_t'(z).$$

Since $\psi_y(T_{gel}) < 1$, Lemma 3.1 implies that X and X' are uniformly bounded in $[0, T_{gel}] \times [0, \psi_y(T_{gel})]$. Hence $\frac{\partial G(t,z)}{\partial z} = \mathcal{O}(1)$, so $G(t,z)$ is Lipschitz in z, uniformly in t. Thus, by Picard's Theorem the equation

$$\frac{d\psi_y(t)}{dt} = \psi_y(t)G(t, \psi_y(t)), \qquad \psi_y(T_{gel}) = \Psi$$

has a unique solution over $[0, T_{gel}]$. Since $G(t,z) \geq 0$ it follows that ψ_y is increasing and since $G(t,z) = 0$ for $z \leq 0$ it follows that $\psi_y(0) > 0$. This completes the construction of the characteristic curves.

By construction ψ_y is continuously differentiable on $(0, T_{gel})$, on (T_{gel}, y) and on (y, ∞). Since it has matching left and right one-sided derivatives at T_{gel} and at y, ψ_y is continuously differentiable on $(0, \infty)$ as required.

To prove the claims about the function $y \mapsto \psi_y(0)$, note first that $\frac{d}{dt}\psi_y(t) \leq \psi_y(t)$ and hence for any $t < y$ we have $1 \geq \psi_y(t) \leq \psi_y(0)e^t$. Letting $t \to y$ we find $\psi_y(0) \geq e^{-y}$, so $0 < \psi_y(0) < 1$ as claimed. The function $y \mapsto \psi_y(0)$ is strictly decreasing since otherwise we would have $y < y'$ such that $\psi_y(0) \geq \psi_{y'}(0)$, in which case there would be some maximal $s < y$ such that $\psi_y(s) = \psi_{y'}(s)$, at which there would be more than one solution to the initial value problem

$$\dot{f}(t) = f(t)\left(1 - X_t(f(t))\right), \quad f(s) = \psi_y(s),$$

contrary to Picard's Theorem.

We now show that $y \mapsto \psi_y(0)$ is a decreasing bijection by exhibiting its inverse. For each $z \in (0,1)$ there exists a unique solution f_z of the initial value problem

$$\dot{f}(t) = f(t)\left(1 - X_t(f(t))\right), \quad f(0) = z$$

taking values in $(0,1)$, on some maximal domain $[0, y(z))$, since the right-hand side of this problem is locally Lipschitz. On the interval $[0, \min(y(z), T_{gel}))$ the same calculations as in Remark 3.7 show that $X_t(f_z(t))$ is constant in t and hence $f_z(t) = ze^{t(1-X_0(z))}$. Since $(1 - X_t(f_z(t))$ is non-negative, the solution is increasing on $[0, y(z))$, so either $f_z(t) \nearrow 1$ as $t \nearrow y(z)$ or $y(z) = \infty$. In the former case the explicit solution on $[0, T_{gel}]$ shows that $y > T_{gel}$, and by the uniqueness proved above we must have $f_z = \psi_{y(z)}$ on $[0, y(z))$. In the latter case we would have $z < \psi_y(0)$ for every $y \in [T_{gel}, \infty)$. Hence if we define $\gamma = \inf\{z \in (0,1) : y(z) < \infty\}$, then we have exhibited an inverse mapping for $y \mapsto \psi_y(0)$, defined on the interval $(z, 1)$. □

Remark 3.9. The method of Lemma 3.5 can also be used to construct the characteristic curves $\xi_s(t)$ defined in equation (66) in Section 3.2 of [24], using the relationship $\xi_s(t) = -\log \psi_s(t)$.

Definition 3.10. *We say that C_t explodes at (the random) time $t \geq 0$ if $C_t = 1$ and for some (random) $\epsilon > 0$ $C_s \neq 1$ for all $s \in (t - \epsilon, t)$.*

An equivalent definition is that C_t explodes at time t if and only if C_t makes infinitely many jumps in (s, t) for every $s < t$. In particular the event that t is an explosion time and the number of explosions that occur in $[0, t]$ are both measurable with respect to $\mathcal{G}_{t-}$, where $\{G_t\}_{t\geq 0}$ is the filtration generated by the càdlàg process C_t.

Lemma 3.11. *For any $y > T_{gel}$ and $0 \leq s < y$, we have*

$$Y_s(\psi_y(s)) = \mathbb{P}[C \text{ does not explode in } [s, y]\,] \tag{3.14}$$

and $\mathbb{P}[C$ explodes at time $y] = 0$. Furthermore,

$$\mathbb{P}[C \text{ explodes during } [0, T_{gel}]\,] = 0\,.$$

Proof. Fix a time $s \geq 0$ and define let $\tau_s = \inf\{t > s : \ C$ explodes at time $t\}$. Note that τ_s is a previsible stopping time. Then the following process is defined for $u \in [s, y)$:

$$M_y(u) = \begin{cases} \psi_y(u)^{C_u} & \text{if } u < \tau_s, \\ 0 & \text{if } u \geq \tau_s. \end{cases}$$

In particular $M_y(s) = \psi_y(s)^{C_s}$, and M_y is adapted to $\{\mathcal{G}_t\}$. We claim that $M_y(\cdot)$ is a $\mathcal{G}_t$-martingale. By conditioning on the first jump in $(t, t + \Delta)$ being of size j, we obtain

$$\begin{aligned} \mathbb{P}[C \text{ jumps at least twice in } (t, t+\Delta)] &= \int_0^\Delta k e^{-sk} \sum_{j=1}^\infty v_j(t+s) \left(1 - e^{-(\Delta - s)(k+j)}\right) ds \\ &= k e^{-k\Delta} \int_0^\Delta 1 - X_{t+s}\left(e^{-(\Delta - s)}\right) ds \\ &\leq k\Delta \sup_{0\leq s\leq \Delta} \left(1 - X_{t+s}\left(e^{-\Delta}\right)\right)\,. \end{aligned}$$

By Dini's theorem, $X_t(e^{-\Delta})$ converges locally uniformly to 1 as $\Delta \searrow 0$, so the last expression above is $o(\Delta)$. It follows that conditional on $C_u = k$ and $u < \tau_s$ the drift of M_y at time u is

$$\begin{aligned} k\psi_y(u)^{k-1} \frac{d}{du}\psi_y(u) + k\sum_{l=1}^\infty (\psi_y(u)^{k+l} - \psi_y(u)^k) v_l(u) \\ = k\psi_y(u)^{k-1}\left(\frac{d}{du}\psi_y(u) + \psi_y(u)\left(X_u(\psi_y(u)) - 1\right)\right) \\ = 0. \end{aligned}$$

That the final line above is 0 follows from Lemma 3.5. Since $M_y(u)$ is bounded, by the martingale convergence theorem we may extend it to a martingale $M_y(u)$ defined for $u \in [s, y]$ that is a.s. continuous at $u = y$.

Note that $C_u \nearrow \infty$ as $u \nearrow \tau_s$. Hence, by Lemma 3.5 if $\tau_s < y$ then $M_y(u) \to 0$ as $u \nearrow \tau_s$ and $M_y(u) = 0$ for $u \in [\tau_s, y)$. If $\tau_s > y$ then $M_y(u) \to 1$ as $u \nearrow y$. If $\tau_s = y$ then all we know is that $M_y(y) \in [0, 1]$. Therefore

$$\mathbf{1}(\tau_s > y) \le M_y(y) \le \mathbf{1}(\tau_s \ge y)\,.$$

Taking conditional expectations on $\mathcal{G}_u$,

$$\mathbb{P}[\tau_s > y \,|\, \mathcal{G}_u] \le M_y(u) \le \mathbb{P}[\tau_s \ge y \,|\, \mathcal{G}_u].$$

Hence, for any $y' > y$, we have

$$\psi_{y'}(s)^{C_s} \le \mathbb{P}[\tau_s \ge y' \,|\, \mathcal{G}_s] \le \mathbb{P}[\tau_s > y \,|\, \mathcal{G}_s] \le \psi_y(s)^{C_s}.$$

Taking expectations of the above equation, we obtain

$$Y_s\left(\psi_{y'}(s)\right) \le \mathbb{P}\left[\tau_s \ge y'\right] \le \mathbb{P}\left[\tau_s > y\right] \le Y_s\left(\psi_y(s)\right).$$

By Lemma 3.5 we can choose y' so as to make $\psi_{y'}(s)$ as close as we like to $\psi_y(s)$, and it follows that $\mathbb{P}[\tau_s = y] = 0$. We obtain also that

$$Y_s\left(\psi_y(s)\right) = \mathbb{P}[C \text{ does not explode in } (s, y)] = \mathbb{P}[C \text{ does not explode in } (s, y]\,]\,. \quad (3.15)$$

Finally, to show that C almost surely does not explode in $[0, T_{gel}]$, by Lemma 3.5 we have $\psi_y(0) \nearrow 1$ as $y \searrow T_{gel}$, and $\lim_{z \nearrow 1} Y_0(z) = 1$, so

$$\begin{aligned}\mathbb{P}[C \text{ does not explode during } [0, T_{gel}]\,] &= \lim_{y \searrow T_{gel}} \mathbb{P}[C \text{ does not explode during } [0, y]\,] \\ &= \lim_{y \searrow T_{gel}} Y_0(\psi_y(0)) \;=\; 1\,.\end{aligned}$$

□

Remark 3.12. By Lemma 3.11, if $\mathbb{P}(C_s = 1) > 0$, which always holds for $s > T_{gel}$, then from (3.15) we obtain

$$\begin{aligned}\psi_y(s) &= \mathbb{P}[C \text{ does not explode in } (s, y) \,|\, C_s = 1] \\ &= \mathbb{P}[C \text{ does not explode in } (s, y] \,|\, C_s = 1]. \end{aligned} \quad (3.16)$$

If $v_1(0) = 0$ then $\mathbb{P}(C_s = 1) = 0$ for $s \le T_{gel}$, in which case conditioning on $C_s = 1$ does not make sense. In this case we can consider a modified version $\hat{C}_t$ of C_t that is started in state 1 at time s, and the same argument shows that (3.16) holds with $\hat{C}$ in place of C.

Corollary 3.13. *Let $y > T_{gel}$. Then $Y_s\left(\psi_y(s)\right) \nearrow 1$ as $s \nearrow y$.*

Proof. This follows from Lemma 3.11 on applying the dominated convergence theorem to the indicator functions $\mathbf{1}(C \text{ does not explode in } [s, y])$ as $s \nearrow y$. □

3.4 Evolution of the watched cluster distribution

We now seek an analogue of equation (1.7) for u_1. For $t \in [0, \infty)$ define

$$\Phi(t) = \mathbb{E}\left[\#\{s \in [0, t) \,:\, C \text{ explodes at time } s\}\right].$$

Recall that C spends an exponential time of mean 1 in state 1 after each explosion. Thus we can stochastically bound the number of explosions in $[0, t]$ by 1 plus a Poisson process

of rate 1, hence $\Phi(t) \le 1 + t$ for all t. Lemma 3.11 implies that Φ is continuous, but we have not yet shown that Φ is differentiable, so we cannot write down a differential equation for u_1. For this reason, it is convenient instead to use integral equations to describe the evolution of (u_k). By examining the transitions of C we obtain

$$u_1(t) = v_1(0) - \int_0^t u_1(s)\,ds + \Phi(t),$$
$$u_k(t) = v_k(0) - \int_0^t k u_k(s)\,ds + \int_0^t \sum_{l=1}^{k-1} l u_l(s) v_{k-l}(s)\,ds \quad \text{for } k \ge 2.$$

Note that the appearance of v_{k-l} corresponds to the fact that each time C jumps it increases by a sample of $l \mapsto v_l(t)$. From the above two equations, for $|z| < 1$ we obtain

$$Y_t(z) = X_0(z) - \int_0^t z Y_s'(z)(1 - X_s(z))\,ds + z\Phi(t)\,. \tag{3.17}$$

Similarly, from (1.7) and Theorem 1.2 we can show that

$$X_s(z) = X_0(z) + \int_0^t z X_s'(z)(1 - X_s(z))\,ds + z\int_0^t \varphi(s)\,ds. \tag{3.18}$$

Combining (3.17) and (3.18) and using the initial condition $Z_0 = Y_0 - X_0 = 0$ we obtain

$$Z_t(z) = zI(t) - \int_0^t z Z_s'(z)(1 - X_s(z))\,ds \tag{3.19}$$

where I is the continuous function defined by

$$I(t) = \Phi(t) - \int_0^t \varphi(s)\,ds.$$

Since the integrand in (3.19) is bounded (by Lemma 3.1), we see that $Z_t(z)$ is continuous in t for each fixed $z \in \mathbb{D}$. Differentiating (3.19) under the integral we find that for $|z| < 1$ we have

$$Z_t'(z) = I(t) - \int_0^t \frac{d}{dz}\left[z Z_s'(z)(1 - X_s(z))\right]\,ds\,. \tag{3.20}$$

To justify this by showing that the integral is absolutely convergent, expand the derivative and apply Lemma 3.1 to bound the result in terms of $|z|$, independently of t. This also shows that for each fixed z with $|z| < 1$, $Z_t'(z)$ is a continuous function of t, and then Lemma 3.1 implies that $Z_t'(s)$ is jointly continuous on $[0,\infty) \times \mathbb{D}$.

For $t \in [0,\infty)$ and $|z| < 1$ define

$$R_t(z) = Z_t(z) - zI(t) = -\int_0^t z Z_s'(z)(1 - X_s(z))\,ds. \tag{3.21}$$

We are aiming to show that both Z and I are identically zero. For each fixed z with $|z| < 1$, we see from the integral expression in (3.21) that $R_t(z)$ is differentiable with respect to both t and z, satisfying

$$\dot{R}_t(z) = -z Z_t'(z)(1 - X_t(z))\,, \tag{3.22}$$
$$R_t'(z) = Z_t'(z) - I(t)\,. \tag{3.23}$$

Hence R' is continuous on $[0,\infty) \times \mathbb{D}$. Using (3.22) with Lemma 3.2 we find also that $\dot{R}_t(z)$ is jointly continuous in t and z and hence $R_t(z)$ is continuously differentiable on $[0,\infty) \times \mathbb{D}$.

Lemma 3.14. *Let $y > T_{gel}$. Then $R_y(\psi_y(t)) \to -I(y)$ as $t \nearrow y$.*

Proof. By definition, $R_y(z) = Y_y(z) - X_y(z) - zI(y)$. By Lemma 3.2 and Lemma 3.11 we know that $X_t(z)$ and $zI(t)$ are both jointly continuous in $(t, z) \in [0, \infty) \times [0, 1]$. From Lemma 3.5 we have $\psi_y(t) \nearrow 1$ as $t \nearrow y$, so $X_t(\psi_y(t)) \to X_y(1) = 1$ as $t \nearrow y$. Corollary 3.13 gives $Y_t(\psi_y(t)) \to 1$ as $t \nearrow y$, so

$$R_y(\psi_y(t)) = Y_y(\psi_y(t)) - X_t(\psi_y(t)) - \psi_y(t)I(t) \to 1 - 1 - I(y)$$

as $t \nearrow y$, as required. □

We are now in a position to prove Proposition 1.9. Recall that Proposition 1.9 stated that for all $t \in [0, \infty)$ and all $l \in \mathbb{N}$, $\mathbb{P}[C_t = l] = v_l(t)$.

Proof of Proposition 1.9. Let $y > T_{gel}$. By Lemma 3.5 we have $\psi_y(t) \in [0, 1)$ for $t < y$. Let

$$\eta_y(t) = R_t(\psi_y(t))$$

for all $y > T_{gel}$ and $t \in [0, y)$.

Combining the continuity and continuous differentiability of $\psi_y(\cdot)$ proved in Lemma 3.5 with the properties of $R_t(z)$ proved above, we find that $t \mapsto \eta_y(t)$ is continuous on $[0, y)$ and continuously differentiable on $(0, y)$. Using (3.22), (3.23) and (3.9) we compute, for $0 < t < y$,

$$\begin{aligned}\frac{d\eta_y(t)}{dt} &= \dot{R}_t(\psi_y(t)) + \frac{d\psi_y(t)}{dt} R'_t(\psi_y(t)))\\ &= -\psi_y(t)Z'_t(\psi_y(t))(1 - X_t(\psi_y(t))) + \psi_y(t)Z'_t(\psi_y(t))(1 - X_t(\psi_y(t))) - \frac{d\psi_y(t)}{dt} I(t)\\ &= -\psi_y(t)(1 - X_t(\psi_y(t))I(t). \qquad (3.24)\end{aligned}$$

Hence for all $t \in [0, y)$ we have

$$\eta_y(t) - \eta_y(0) = -\int_0^t \psi_y(s)(1 - X_s(\psi_y(s)))I(s)ds.$$

Using (3.21) and Lemma 3.14 we have $\eta_y(0) = R_0(\psi_y(0)) = 0$ and $\lim_{t \nearrow y} \eta_y(t) = -I(y)$. Hence

$$I(y) = \int_0^y \psi_y(s)(1 - X_s(\psi_y(s)))I(s)ds$$

which implies that

$$|I(y)| \leq \int_0^y |I(s)|ds.$$

The above equation holds for all $y > T_{gel}$. By Lemma 3.11 and (1.7) we have $I(y) = 0$ for all $y < T_{gel}$. Using Grönwall's inequality, this shows that I is identically zero. Hence from (3.24) we have $\frac{d\eta_y(t)}{dt} = 0$ and since $\eta_y(0) = 0$ we have $\eta_y = 0$. Hence, from (3.21), for all $y > T_{gel}$ and $t \in [0, y)$ we have $Z_t(z) = zI(t) = 0$ for all $z \in [\psi_y(t), 1)$. According to Lemma 3.5, we have $\psi_y(t) < 1$, so by the identity theorem this implies that Z_t is identically zero for each $t < y$. Since $y > T_{gel}$ was arbitrary, this shows that for every $t \in [0, \infty)$ we have $Y_t = X_t$ and hence $u_k(t) = v_k(t)$ for all $k \in \mathbb{N}$. □

3.5 C_t almost surely explodes infinitely often

We are now in a position to establish an important property of C, namely that it explodes infinitely often.

Lemma 3.15. *For all $t \geq 0$,*

$$\mathbb{E}(1/C_t) = \mathbb{E}(1/C_0) + \int_0^t (\varphi(s) - 1/2)\, ds\,. \tag{3.25}$$

Proof. By Proposition 1.9 and Fubini's theorem,

$$\mathbb{E}(1/C_t) = \sum_{k=1}^{\infty} \frac{v_k(t)}{k} = \int_0^1 \frac{X_t(z)}{z}\, dz\,.$$

Hence, using (3.3) and Fubini's theorem again,

$$\begin{aligned}
\mathbb{E}(1/C_t) - \mathbb{E}(1/C_0) &= \int_0^1 \frac{X_t(z) - X_0(z)}{z}\, dz \;=\; \int_0^1 \int_0^t \frac{\dot{X}_t(z)}{z}\, dt\, dz \\
&= \int_0^1 \int_0^t X_t'(z)\,(X_t(z) - 1) + \varphi(t)\, dt\, dz \\
&= \int_0^t \int_0^1 \frac{d}{dz}\left(\frac{X_t(z)^2}{2} - X_t(z)\right) dz + \varphi(t)\, dt\,,
\end{aligned}$$

This gives the desired result since $X_t(1) = 1$ and $X_t(0) = 0$ for all t. □

Lemma 3.16. *For every $t \geq 0$, $\lim_{y\to\infty} \psi_y(t) = 0$. The characteristic curves $\psi_\cdot(\cdot)$ fill $[0,\infty) \times (0,1)$ and C_t almost surely explodes infinitely often.*

Proof. For any $y > T_{gel}$ and $0 \leq t \leq s < y$ we have

$$1 \;\geq\; X_s\left(\psi_y(s)\right) \;=\; X_t\left(\psi_y(t)\right) + \int_t^s \psi_y(u)\varphi(u)\, du \;\geq\; \psi_y(t) \int_t^s \varphi(u)\, du\,.$$

By Lemma 3.15 we have

$$\int_t^s \varphi(u)\, du = \frac{s-t}{2} + \mathbb{E}\left(1/C_s\right) - \mathbb{E}\left(1/C_t\right) \geq \frac{s-t}{2} - 1\,,$$

so letting $s \nearrow y$ we obtain

$$0 \leq \psi_y(t) \leq \frac{1}{\frac{y-t}{2} - 1}\,.$$

Hence for every $t \geq 0$ and $0 < z < 1$ we can find y large enough to ensure $\psi_y(t) < z$. Now it follows by the arguments used to prove Lemma 3.5 that there exists $y > T_{gel}$ such that $\psi_y(t) = z$.

To conclude that C_t almost surely explodes infinitely often, we use Lemma 3.11, to see that

$$\mathbb{P}(C \text{ does not explode after time } t) = \lim_{y\to\infty} Y_t(\psi_y(t)) = Y_t(0) = 0$$

as required. □

Remark 3.17. Note that, since $\frac{1}{C_t} \in (0,1]$, (3.25) establishes a weak sense in which $\varphi(t)$ approaches $\frac{1}{2}$ as $t \to \infty$. We conjecture that, in fact, $\varphi(t) \to \frac{1}{2}$ as $t \to \infty$.

4 Coupling

In this section we will prove Theorem 1.7 by coupling the pair (C^n, C) so that, for any fixed $T > 0$, if we take n sufficiently large then, with probability close to 1 we have $C^n(\cdot) = C(\cdot)$ for all but a small proportion of the time interval $[0, T]$ and at the exceptional times $C^n(\cdot)$ and $C(\cdot)$ are nevertheless close in the compact state space E. The coupling divides into two parts:

1. If C and C^n are small, and equal, in size then their respective jump rates are exponential random variables of similar rate and we can use Theorem 1.5 to couple the size of the corresponding jumps. Note that this incurs a small probability of failure, caused by the jump rates not quite being equal and by v^n and v (which define the jump distributions) being not quite equal.
2. Eventually C and C^n become large enough that the probability of failure incurred above is too high to control. At this point we rely on conservativity (1.6), which combined with Proposition 1.9 and Theorem 1.5 implies that a cluster which is already large will burn quickly (in both C and C^n). Once they burn, with high probability they stay in state 1 for long enough to enable recoupling and we can once again use Theorem 1.5.

We now proceed with the argument, which is rather involved. Our first step is to show that large clusters burn quickly and once that is done we will construct the coupling.

Lemma 4.1. *Let $\epsilon > 0$. Let $T > 0$ and $\delta \in (0, T)$. Then there exists $K, N \in \mathbb{N}$ such that for all $n \geq N$ and all $k \geq K$,*

$$\mathbb{P}\left[\exists t \in [\delta, T] \text{ such that } \inf_{s\in[t-\delta,t]} C_s^n > k\right] < \epsilon.$$

Proof. Let $T, \epsilon > 0$, $\delta \in (0, T)$ and $A_{k,\delta} = \left\{\exists t \in [\delta, T] \text{ such that } \inf_{s\in[t-\delta,t]} C_s^n > k\right\}$. On the event $A_{k,\delta}$ we have $\int_0^T \mathbb{1}\{C_s^n > k\}ds \geq \delta$ and thus

$$\delta\, \mathbb{P}\left[A_{k,\delta}\right] \leq \mathbb{E}\left[\int_0^T \mathbb{1}\{C_s^n > k\}ds\right]. \tag{4.1}$$

We now seek an upper bound on the right hand side of (4.1). Since our tagged vertex p was sampled uniformly from $[n]$ we have $P\left[C_s^n = j\right] = \mathbb{E}\left[v_j^n(s)\right]$. Hence,

$$\begin{aligned}\mathbb{E}\left[\int_0^T \mathbb{1}\{C_s^n < k\}\, ds\}\right] &= T - \mathbb{E}\left[\int_0^T \mathbb{1}\{C_s^n \leq k\}\, ds\right] \\ &= T - \sum_{j=1}^{k} \mathbb{E}\left[\int_0^T v_j^n(s)\, ds\right]. \end{aligned} \tag{4.2}$$

By Theorem 1.5 and the fact that $v_j^n(t) \in [0, 1]$, for each $j \in \mathbb{N}$ we have

$$\mathbb{E}\left[\int_0^T v_j^n(s)\, ds\right] \to \int_0^T v_j(s)\, ds \quad \text{as } n \to \infty\,. \tag{4.3}$$

Since $\sum_{j=1}^{\infty} v_j(s) = 1$ we have $T = \int_0^T \sum_{j=1}^{\infty} v_k(s)\, ds = \sum_{j=1}^{\infty} \int_0^T v_k(s)\, ds$. Hence we can choose $K \in \mathbb{N}$ such that for all $k \geq K$,

$$T - \sum_{j=1}^{k} \int_0^T v_j(s)\, ds < \frac{\epsilon\delta}{2}. \tag{4.4}$$

Using (4.3), we choose $N \in \mathbb{N}$ such that for all $j = 1, \dots, K$ and $n \geq N$, we have

$$\left| \mathbb{E}\left[\int_0^T v_j^n(s)\, ds \right] - \int_0^T v_j(s)\, ds \right| < \frac{\epsilon\delta}{2K}.$$

Putting the above equation and (4.4) into (4.2) we get

$$\mathbb{E}\left[\int_0^T \mathbb{1}\{C_s^n < k\}\, ds \right] < \epsilon\delta.$$

Combining this equation with (4.1) obtains the stated result. □

Lemma 4.2. *Let $\epsilon > 0$. Let $T > 0$ and $\delta \in (0, T)$. Then there exists $K \in \mathbb{N}$ such that for all $k \geq K$,*

$$\mathbb{P}\left[\exists t \in [\delta, T] \text{ such that } \inf_{s \in [t-\delta, t]} C_s > k \right] < \epsilon.$$

Proof. Let $\epsilon, T > 0$ and $\delta \in [0, T]$. By Proposition 1.9 we have $\mathbb{P}[C_s = j] = v_j$ for all s. We set $A_{k,\delta} = \left\{ \exists t \in [\delta, T] \text{ such that } \inf_{s \in [t-\delta, t]} C_s > k \right\}$ and then, in similar style to (4.1) and (4.2), we obtain

$$\delta\, \mathbb{P}[A_{k,\delta}] \leq \mathbb{E}\left[\int_0^T \mathbb{1}\{C_s > k\}\, ds \right] = T - \sum_{j=1}^{k} \int_0^T v_j(s)\, ds.$$

By (1.6) we have $T = \sum_{j=1}^{\infty} \int_0^T v_j(s) ds$ so we can choose $K \in \mathbb{N}$ such that for all $k \geq K$,

$$T - \sum_{j=1}^{k} \int_0^T v_j(s) ds < \epsilon\delta$$

and we are done. □

Our next step is to construct a coupling between C and C^n (where n is still to be chosen). In fact we will define a càdlàg E valued process $\widetilde{C}$ which has the same distribution as C and is coupled to C^n. Our coupling will be such that with probability close to 1 the distance $d_E(\widetilde{C}_t, C_t^n)$ remains small up until a given large time T. We will also define a process S, taking values in $\{\mathbf{0}, \mathbf{1}\}$. The process S acts as a *state bit*; for as long as we can keep $\widetilde{C}$ and C^n close we have $S = \mathbf{0}$. The time

$$\tau = \inf\{s > 0 \,:\, S_s = \mathbf{1}\}$$

records the time at which our coupling fails to keep $\widetilde{C}$ and C^n close.

At this point we make a minor modification to our model $\mathcal{Z}^n$:

(†) Growth clocks corresponding to (unordered) pairs (i, j) where $i, j \in \mathcal{C}^n$ and $i \neq j$ ring at twice their normal rate (i.e. at rate $\frac{2}{n}$ instead of $\frac{1}{n}$).

This modification has no effect on the clusters which form in $\mathcal{Z}^n$, since any such pair (i, j) were already part of the same connected cluster. We may assume (†) without any change to the dynamics of the partition of vertices into clusters. We are not interested here in the internal structure of the clusters, so we assume (†) from now on. Thanks to (†), we obtain the following properties:

(A) Given that $C_t^n = k$, the time until one of the growth clocks of some (i, j) with $i \in \mathcal{C}_t^n$ next rings is exponential with rate k.

(B) When the corresponding edge appears, at time t', the effect is that a vertex i is sampled uniformly from $\mathcal{C}^n_{t'-}$ and a second vertex j is sampled uniformly from $[n]$ (and an edge is created between these two vertices).

Remark 4.3. Without (†), given $C^n_t = k$, the time until one of the growth clocks of some (i,j) with $i \in \mathcal{C}^n_t$ rings would be exponential with rate

$$R^n_k = \frac{1}{n}\left(k(n-k) + k + \binom{k}{2}\right). \tag{4.5}$$

This does not match the jump rate of $\widetilde{C}^n$, causing additional error terms that we would then need to control (using that fact that $R^n_k \to k$ as $n \to \infty$). Further, without (†) we would not have property (B) and this would create yet more error terms.

Let $H(i,j)$ denote the growth clock for the (unordered) pair (i,j). Let $I_{0,0} = 0$ and for $a \geq 0$ define inductively

$$I_{a+1,0} = \inf\{s > I_{a,0} \,:\, \mathcal{C}^n \text{ is burned at time } s\}.$$

Thus $I_{a,0}$ is the time of the a^{th} fire of $\mathcal{C}^n$. Define g_a to be the number of growth events of $\mathcal{C}^n$ between $I_{a,0}$ and $I_{a+1,0}$. Now define inductively for $a \geq 0$ and $0 \leq b < g_a$

$$I_{a,b+1} = \inf\{s > I_{a,b} \,:\, s < I_{a+1,0} \text{ and } \exists i \in \mathcal{C}^n_{s-}, j \in \mathbb{N} \text{ such that } H(i,j) \text{ rings at } s\}.$$

Thus, $I_{a,b}$ is the b^{th} time after the a^{th} fire of $\mathcal{C}^n$ at which a growth clock of some (i,j) with at least one of i and j currently in $\mathcal{C}^n$ rings. Note that the (random) set

$$\mathscr{I} = \{I_{a,b} \,:\, a \geq 0,\ 0 \leq b \leq g_a\}$$

is a well–ordered subset of $[0,\infty)$ and its order agrees with the lexicographic ordering of the corresponding indices (a,b). With a slight abuse of language, we say that $I_{a,b} \in \mathscr{I}$ is a fire if $b = 0$ and growth if $b > 0$.

Consider the time $I_{a,b}$, for $b \geq 1$, corresponding to, say, the growth clock $H(i_{a,b}, j_{a,b})$ where by property (B) we have that (conditionally on $\mathcal{Z}^n$ prior to $I_{a,b}$, and independently of all else) $i_{a,b}$ is sampled uniformly from $\mathcal{C}^n_{I_{a,b}-}$ and $j_{a,b}$ is sampled uniformly from $[n]$. In fact we perform the sampling of $j_{a,b}$ using a paintbox-type construction. It is advantageous to introduce some extra notation to explain this. We recall that $\mathcal{Z}^n$ is invariant under permutations of the labels $[n]$ of the vertices, so (for convenience) at all times we label the vertices in increasing order of cluster size. To sample $j_{a,b}$, we sample a uniform random variable $U_{a,b}$ on $[0,1]$, independently of all else, and we set

$$j_{a,b} = \min\left\{j' \in [n] \,:\, U_{a,b} < \frac{j'}{n}\right\}.$$

Because the vertices are labelled in order of their cluster size, the cluster size

$$L_{a,b} = C^n_{I_{a,b}-}(j_{a,b})$$

satisfies

$$\sum_{l=1}^{L_{a,b}} v^n_l(I_{a,b}-) \leq U_{a,b} < \sum_{l=1}^{L_{a,b}+1} v^n_l(I_{a,b}-), \tag{4.6}$$

which simply states that $C^n_{I_{a,b}-}(j_{a,b})$ has conditional distribution $l \mapsto v^n_l(I_{a,b}-)$. We additionally sample a random variable $\widetilde{L}_{a,b}$ given by

$$\sum_{l=1}^{\widetilde{L}_{a,b}} v_l(I_{a,b}-) \leq U_{a,b} < \sum_{l=1}^{\widetilde{L}_{a,b}+1} v_l(I_{a,b}-). \tag{4.7}$$

Thus $\widetilde{L}_{a,b}$ has conditional distribution $l \mapsto v_l(I_{a,b})$.

For any $t \in [0,\infty)$ we define

$$t^{\oplus} = \min\{s \in \mathscr{I} : s > t\}.$$

Note that (almost surely) each time $t^{\oplus}$ is either a fire or a growth.

Let $K \in \mathbb{N}$ be arbitrary, for now; it will be given a fixed value later. We partition $\{\mathbf{0},\mathbf{1}\} \times E \times E$ into six subsets, corresponding to six cases in the definition of the joint evolution of $(S_t, C_t^n, \widetilde{C}_t)$. These are

$$\begin{aligned}
E_1 &= \{\mathbf{0}\} \times \{(k,k) : k \le K\} \\
E_2 &= \{\mathbf{0}\} \times \{(k,\overline{k}) : k,\overline{k} > K\} \\
E_3 &= \{\mathbf{0}\} \times \{(k,1) : k > K\} \\
E_4 &= \{\mathbf{0}\} \times \{(1,\overline{k}) : \overline{k} > K\} \\
E_5 &= \{\mathbf{1}\} \times E \times E \\
E_6 &= (\{\mathbf{0},\mathbf{1}\} \times E \times E) \setminus \cup_{m=1}^{5} E_m.
\end{aligned}$$

In similar style to (4.6), at time 0 we sample a uniform random variable U on $[0,1]$, take the watched point p to be $p = \min\{p' \in [n] : U < \frac{p'}{n}\}$ and take the size C_0^n of the watched cluster to be

$$\sum_{l=1}^{C_0^n} v_l^n(0) \le U < \sum_{l=1}^{C_0^n+1} v_l^n(0). \tag{4.8}$$

Similarly, we define the initial state of $\widetilde{C}$ by

$$\sum_{l=1}^{\widetilde{C}_0} v_l(0) \le U < \sum_{l=1}^{\widetilde{C}_0+1} v_l(t). \tag{4.9}$$

If $C_0^n = \widetilde{C}_0$ then we set $S_0 = \mathbf{0}$, otherwise we set $S_0 = \mathbf{1}$. The evolution of $(S_t, C_t^n, \widetilde{C}_t)$ then proceeds as follows; the infinitesimal evolution is different depending on which E_i the process $(S, C^n, \widetilde{C})$ is in.

- If $(S_t, C_t^n, \widetilde{C}_t) = (\mathbf{0}, k, k) \in E_1$ then, at time $t^{\oplus}$,
  - if $t^{\oplus} = I_{a,b}$ is a growth, and both $j_{a,b} \notin \mathcal{C}_{t^{\oplus}}^n$ and $L_{a,b} = \widetilde{L}_{a,b}$, it jumps to $(\mathbf{0}, C_t^n + L_{a,b}, \widetilde{C}_t + \widetilde{L}_{a,b})$.
  - if $t^{\oplus} = I_{a,b}$ is a growth, and both $j_{a,b} \notin \mathcal{C}_{t^{\oplus}}^n$ and $L_{a,b} \neq \widetilde{L}_{a,b}$, it jumps to $(\mathbf{1}, C_t^n + L_{a,b}, \widetilde{C}_t + \widetilde{L}_{a,b})$.
  - if $t^{\oplus} = I_{a,b}$ is a growth, and $j_{a,b} \in \mathcal{C}_{t^{\oplus}}^n$, it jumps to $(\mathbf{1}, C_t^n, \widetilde{C}_t + \widetilde{L}_{a,b})$.
  - if $t^{\oplus}$ is a fire, it jumps to $(\mathbf{1}, 1, \widetilde{C}_t)$.
- While $(S, C^n, \widetilde{C}) = (\mathbf{0}, k, \overline{k}) \in E_2$, the processes C^n and $\widetilde{C}^n$ evolve independently of one another. The evolution of C^n is already specified, S remains constant at $\mathbf{0}$ and the evolution of $\widetilde{C}$ is that
  - if $\widetilde{C}_t = k$, then at rate k it jumps to $\widetilde{C}_t + L$, where L is sampled from $v_l(\alpha)$ and α is the jump time.

  Note that this could result in infinitely many jumps of $\widetilde{C}$ inside E_2, in which case, by definition of E, at the time of the accumulation point of these jumps $\widetilde{C}$ enters state 1.
- While $(S_t, C_t^n, \widetilde{C}_t) = (\mathbf{0}, k, 1) \in E_3$ then,

  - At rate 1, it jumps to $(\mathbf{1}, C^n_t, \widetilde{C}_t + L)$, where L is sampled from $v_l(\alpha)$ and α is the jump time.

  If $t^\oplus$ occurs before this (potential) jump then, at time $t^\oplus$,

  - if $t^\oplus = I_{a,b}$ is a growth, it jumps to $(\mathbf{0}, C^n_t + L_{a,b}\mathbb{1}\{j_{a,b} \notin \mathcal{C}^n_{t^\oplus}\}, 1)$.
  - if $t^\oplus$ is a fire, it jumps to $(\mathbf{0}, 1, 1)$.

- While $(S_t, C^n_t, \widetilde{C}_t) = (\mathbf{0}, 1, \overline{k}) \in E_4$, the processes C^n and $\widetilde{C}^n$ evolve independently of one another. The evolution of $\widetilde{C}$ is that

  - if $\widetilde{C}_t = k$, then at rate k it jumps to $\widetilde{C}_t + L$, where L is sampled from $v_l(\alpha)$ and α is the jump time. In this case S and C^n remain constant.

  If $t^\oplus$ occurs before this (potential) jump then, at time $t^\oplus$,

  - if $t^\oplus = I_{a,b}$ is a growth, it jumps to $(\mathbf{1}, 1 + L_{a,b}\mathbb{1}\{j_{a,b} \notin \mathcal{C}^n_{t^\oplus}\}, \widetilde{C}_t)$.
  - if $t^\oplus$ is a fire, there is no change.

- While $(S, C^n, \widetilde{C}) \in E_5$, we have $S = \mathbf{1}$ and the processes C^n and $\widetilde{C}$ evolve independently of one another. The evolution of C^n is already specified and the evolution of $\widetilde{C}$ is the same as defined above in the case of E_2.
- The process $(S, C^n, \widetilde{C})$ does not enter E_6.

Remark 4.4. For as long as $S = \mathbf{0}$, the transitions between the E_i of $(S, C^n, \widetilde{C})$ follow the cycle $E_1 \to E_2 \to (E_3 \cup E_4) \to E_1$. Precisely one of E_3 and E_4 is visited in each such cycle. Once $(S, C^n, \widetilde{C})$ has entered E_5 (which happens as soon as $S = \mathbf{1}$), it never leaves.

By comparing each case in turn, it can be seen that the evolution specified for C^n matches that given in Section 1. Further, it is clear from the above definition that the càdlàg process $(S, C^n, \widetilde{C})$ is strongly Markov with respect to its generated filtration $(\mathcal{F}_t)$. Note that this filtration is the product of the filtration of $\mathcal{Z}^n$ and the additional randomness introduced above (i.e. $\widetilde{C}$, $U_{a,b}$, $L_{a,b}$ and so on). With mild abuse of notation, we extend our probability measure $\mathbb{P}$ to be a measure on $\sigma(\cup_{t\in[0,\infty)}\mathcal{F}_t)$.

Lemma 4.5. *The processes $\widetilde{C}$ and C have the same distribution.*

Proof. When $(S, C^n, \widetilde{C}) \in E_2 \cup E_4 \cup E_5$, the evolution of $\widetilde{C}$ defined above is trivially the same as that given in Definition 1.6. If $(S, C^n, \widetilde{C}) \in E_1$, then $\widetilde{C}$ jumps at rate k (corresponding to the next $t^\oplus$ that is a growth). On such a jump at time, say α, the jump causes a displacement L with distribution (conditional on α) given by $l \mapsto v_l(\alpha)$. If $(S, C^n, \widetilde{C}) \in E_3$ then $\widetilde{C} = 1$ and in this case jumps of $\widetilde{C}$ occur at rate one, with the jump distribution $l \mapsto v_l(\alpha)$.

Thus, in all cases $\widetilde{C}$ has the same jump rates and jump distributions as C. Since the paths of C and $\widetilde{C}$ are characterized entirely by their jump times and corresponding displacements, C and $\widetilde{C}$ have equal distribution. □

Remark 4.6. The process $\widetilde{C}$ clearly depends on n. However, it follows from Lemma 4.5 that the distribution of $\widetilde{C}$ does not depend on n. It is for this reason that we choose not to add a superscript n onto $U_{a,b}$, $L_{a,b}$, etc.

We now aim to show that, given $T \in (0, \infty)$, K and n can be chosen so that $\mathbb{P}[\tau \le T]$ is arbitrarily small. Our first step, which will allow us to make use of (4.6), (4.7), (4.8) and (4.9), is the following lemma.

Lemma 4.7. *Let $(x_n)_{n=1}^{\infty}, (x'_n)_{n=1}^{\infty} \subseteq [0,1]$ be random sequences such that $\sum_n x_n = \sum x'_n = 1$. Let $U \in [0,1]$ be a uniform random variable on $[0,1]$ which is independent of (x_n) and (x'_n). Define $c, c' \in \mathbb{N}$ by*

$$\sum_{k=1}^{c} x_k \leq U < \sum_{k=1}^{c+1} x'_k, \qquad \sum_{k=1}^{c'} x'_k \leq U < \sum_{k=1}^{c'+1} x'_k. \tag{4.10}$$

Suppose that $\eta > 0$ is such that

$$\mathbb{P}\left[\exists k \leq K, |x_k - x'_k| \geq \frac{\eta}{K^2}\right] \leq \frac{\eta}{K}, \qquad \mathbb{P}\left[\sum_{k>K} x_k \leq \eta\right] = 1. \tag{4.11}$$

Then $\mathbb{P}[c = c'] \geq 1 - 6\eta$.

Proof. We note that

$$\sum_{k>K} x'_k = \sum_{k>K} x_k + \sum_{k=1}^{K} (x_k - x'_k)$$

so that (in similar style to (2.10)) from (4.11) we have

$$\mathbb{P}\left[\sum_{k>K} x'_k > 2\eta\right] \leq \eta. \tag{4.12}$$

Writing $s_k = \sum_{l=1}^{k} x_k$ and $s'_k = \sum_{l=1}^{k} x'_k$, from (4.10) we have that

$$\{c \neq c'\} \subseteq \left\{U \geq \sum_{k=1}^{K} x_k\right\} \cup \left\{U \geq \sum_{k=1}^{K} x'_k\right\} \cup \left(\bigcup_{k=1}^{K} \{U \in [\min(s_k, s'_k), \max(s_k, s'_k)]\}\right). \tag{4.13}$$

Using that

$$|\max(s_k, s'_k) - \min(s_k, s'_k)| \leq \sum_{l=1}^{k} |x_k - x'_k| \leq \sum_{l=1}^{K} |x_k - x'_k|$$

in (4.13), along with (4.11) and (4.12), we obtain

$$\mathbb{P}[c \neq c'] \leq \eta + 3\eta + \frac{\eta}{K} + \sum_{k=1}^{K} \frac{\eta}{K} \leq 6\eta$$

as required. □

Let $\epsilon > 0$, let $T \in (0, \infty)$. Let $J \in \mathbb{N}$ be large enough that

$$\mathbb{P}[I_{J+1,0} \leq T] < \epsilon. \tag{4.14}$$

Let $\delta > 0$ be such that

$$\delta J \leq \epsilon. \tag{4.15}$$

Note that it is trivial that such a J exists, since C^n spends an exponential time of rate 1 at state 1 upon each return. We now choose the value of K. First, using the definition of E and Assumption 1.1, let $K, N \in \mathbb{N}$ be large enough that, for all $n \geq N$,

$$d(1, K) \leq \epsilon, \tag{4.16}$$

$$K\lambda_n T \leq \epsilon, \tag{4.17}$$

$$\frac{K^2 J}{N} \leq \epsilon \tag{4.18}$$

Then, by Theorem 1.5, combined with Lemmas 4.1 and 4.2 (the latter of which applies by Lemma 4.5), we may increase K and N so that for all $n \geq N$,

$$\mathbb{P}\left[\exists t \in [\delta, T] \text{ such that } \inf_{s\in[t-\delta,t]} C^n_s > K\right] \leq \frac{\epsilon}{J}, \tag{4.19}$$

$$\mathbb{P}\left[\exists t \in [\delta, T] \text{ such that } \inf_{s\in[t-\delta,t]} \widetilde{C}_s > K\right] \leq \frac{\epsilon}{J}, \tag{4.20}$$

$$\mathbb{P}\left[\sup_{l\in\mathbb{N}} \sup_{s\in[0,T]} |v^n_l(t) - v_l(t)| > \frac{\epsilon}{K^3 J}\right] \leq \frac{\epsilon}{K^2 J}. \tag{4.21}$$

Using Dini's Theorem, we may increase K so that also

$$\sup_{s\in[0,T]} \sum_{k>K} v_k(s) < \epsilon. \tag{4.22}$$

Finally we may increase N if necessary to ensure that inequalities (4.16)–(4.22) hold simultaneously.

Lemma 4.8. *If $S_t = \mathbf{0}$ then $d_E(C^n_t, \widetilde{C}_t) \leq \epsilon$.*

Proof. This follows immediately from (4.16), the definition of $(S, C^n, \widetilde{C})$ and the definition of d_E. □

Lemma 4.9. *It holds that $\mathbb{P}[\tau \leq T] \geq 1 - 22\epsilon$.*

Proof. Using the definition of the tagged vertex p, (4.8), (4.9), (4.21) and (4.22), we can apply Lemma 4.7 to C^n_0 and C (with $\eta = \epsilon$, $x_k = v_k(0)$ and $x'_k = v^n_k(0)$) and obtain that

$$\mathbb{P}\left[C^n_0 \neq \widetilde{C}_0\right] \leq 6\epsilon. \tag{4.23}$$

In view of the above equation and Lemma 4.8, in order to prove the current lemma we must control the probabilities of S exiting state $\mathbf{0}$ during time $[0, T]$. This exit can occur in several different ways, as can be seen from the definition of $(S, C^n, \widetilde{C})$. We go through each possible case (in the same order as they occur within the definition of $(S, C^n, \widetilde{C})$) and establish a bound on the probability of each. Let us first examine the transitions out of E_1 that can lead to $S = \mathbf{1}$.

- When C^n makes a jump from a state in $\{1, \ldots, K\}$ at time $I_{a,b}$, the probability that this jump has $L_{a,b} \neq \widetilde{L}_{a,b}$ can be bounded above using Lemma 4.7. Using, (4.6), (4.7), (4.21) and (4.22) (with $\eta = \frac{\epsilon}{KJ}$), we obtain that this probability is bounded above by $\frac{6\epsilon}{JK}$. Now, by (4.14), with probability at least $1 - \epsilon$ the process C^n exits state 1 at most J times during $[0, T]$. On this event, there can be at most JK jumps of C^n from states in $\{1, \ldots, K\}$ during $[0, T]$. Hence,

$$\mathbb{P}\left[\exists\, a, b \text{ such that } I_{a,b} \leq T \text{ and } C_{I_{a,b}-} \leq K \text{ and } L_{a,b} \neq \widetilde{L}_{a,b}\right] \leq \epsilon + KJ\frac{6\epsilon}{JK} = 7\epsilon. \tag{4.24}$$

- When C^n makes a jump from a state in $\{1, \ldots, K\}$ at time $I_{a,b}$, the probability that this jump has $j_{a,b} \in \mathcal{C}^n_{I_{a,b}}$ is equal to the probability that a uniform random element of $\{1, \ldots n\}$ is within $\{1, \ldots K\}$, which is itself $\frac{K}{n}$. As in the above case, by (4.14), with probability at least $1 - \epsilon$ there are at most JK such jumps. Hence, by (4.18)

$$\mathbb{P}\left[\exists\, a, b \text{ such that } I_{a,b} \leq T \text{ and } j_{a,b} \in \mathcal{C}^n_{I_{a,b}}\right] \leq \epsilon + JK\frac{K}{N} \leq 2\epsilon. \tag{4.25}$$

- Given that $C^n = k$, the rate at which $\mathcal{C}^n$ burns is $k\lambda_n$. Hence, the probability that $\mathcal{C}^n$ sees a fire at some time $t \in [0, T]$ for which $C^n_t \leq K$ is bounded above by $1 - e^{K\lambda_n T}$. Hence, by (4.17) we have that

$$\mathbb{P}\left[\exists t \in [0,T] \text{ such that } t^{\oplus} \text{ is a fire and } C^n_t \leq K\right] \leq \epsilon. \tag{4.26}$$

There are no transitions out of E_2 that can lead to $S_t = \mathbf{1}$. We now move on to the transitions out of $E_3 \cup E_4$ that can lead to $S = \mathbf{1}$.

- The only possible transition out of E_3 which leads to $S = \mathbf{1}$ is if $\widetilde{C}$ makes a jump. This occurs at rate 1. By (4.19) combined with (4.14), with probability $1 - 2\epsilon$, the total time spent in E_3 is at most $J\delta$. Thus, using also (4.15) we have

$$\begin{aligned}\mathbb{P}\left[\exists t \in [0,T] \text{ such that } (S_t, C^n_t, \widetilde{C}_t) \in E_3 \text{ and } \widetilde{C} \text{ jumps during } [t, t^{\oplus}]\right] \\ \leq 2\epsilon + (1 - e^{-J\delta}) \\ \leq 3\epsilon. \end{aligned} \tag{4.27}$$

- The only possible transition out of E_4 which leads to $S = \mathbf{1}$ is if C^n makes a jump. This is essentially the same case as E_3 (see above), except that the roles of C^n and $\widetilde{C}$ are reversed. We apply the same argument as is used above, with (4.20) replacing 4.19, to obtain

$$\mathbb{P}\left[\exists t \in [0,T] \text{ such that } (S_t, C^n_t, \widetilde{C}_t) \in E_4 \text{ and } C^n \text{ jumps before } \widetilde{C}\right] \leq 3\epsilon. \tag{4.28}$$

Since $S = \mathbf{1}$ within E_5, and E_6 is never visited, there are no more jumps in which the value of S can change from $\mathbf{0}$ to $\mathbf{1}$. Summing up our error terms in (4.23)-(4.28), we obtain the required result. □

Remark 4.10. It is clear from the proof that of Lemma 4.9 that, during $[0, \tau)$, the process $(S, C^n, \widetilde{C})$ spends most of its time in E_1, during which $C^n = \widetilde{C}$.

Proof of Theorem 1.7. By Lemma 4.5, the $E \times E$ valued process $(C^n, \widetilde{C})$ is a coupling of C^n and C. By Lemmas 4.8 and 4.9 we have $\mathbb{P}[\sup_{t \leq T} d_E(\widetilde{C}, C^n) > \epsilon] \leq \mathbb{P}[\tau \geq T] \leq 22\epsilon$ for all $n \geq N$. □

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